Diagonals of Certain Operators
in von Neumann Algebras

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Joint work with Matt Kennedy

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Diagonals in a Von Neumann Algebras

Diagonal of a Matrix

Let $D_n$ be the diagonal subalgebra of $M_n(\mathbb{C})$ and let $E_{D_n} : M_n(\mathbb{C}) \to D_n$ be defined by

$$E_{D_n}([a_{i,j}]) = (a_{1,1}, a_{2,2}, \ldots, a_{n,n}).$$

Given $T \in M_n(\mathbb{C})$ with fixed properties, what values can $E_{D_n}(T)$ take?
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**Diagonal of a Matrix**

Let $\mathcal{D}_n$ be the diagonal subalgebra of $\mathcal{M}_n(\mathbb{C})$ and let $E_{\mathcal{D}_n} : \mathcal{M}_n(\mathbb{C}) \to \mathcal{D}_n$ be defined by

$$E_{\mathcal{D}_n}([a_{i,j}]) = (a_{1,1}, a_{2,2}, \ldots, a_{n,n}).$$

Given $T \in \mathcal{M}_n(\mathbb{C})$ with fixed properties, what values can $E_{\mathcal{D}_n}(T)$ take?

**Analogue in von Neumann Algebras**

Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{A}$ be a MASA in $\mathcal{M}$, and let

$$E_\mathcal{A} : \mathcal{M} \to \mathcal{A}$$

be a conditional expectation of $\mathcal{M}$ onto $\mathcal{A}$. Given $T \in \mathcal{M}$ with fixed properties, what operators may $E_\mathcal{A}(T)$ be?
The Schur-Horn Theorem

**Theorem (Schur; 1923)**

If \( T \in \mathcal{M}_n(\mathbb{C}) \) is a self-adjoint matrix with eigenvalues and diagonal entries

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad a_1 \geq a_2 \geq \cdots \geq a_n
\]

respectively, then

1. \( \sum_{k=1}^{m} a_k \leq \sum_{k=1}^{m} \lambda_k \) for all \( m \in \{1, \ldots, n\} \), and
2. \( \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \lambda_k \).

**Theorem (Horn; 1954)**

If \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad a_1 \geq a_2 \geq \cdots \geq a_n \)

are elements of \( \mathbb{R} \) such that the above two conditions hold, then there exists a self-adjoint matrix \( T \in \mathcal{M}_n(\mathbb{C}) \) with eigenvalues \((\lambda_k)_{k=1}^{n}\) and diagonal entries \((a_k)_{k=1}^{n}\).
The Carpenter’s Theorem

In the beautiful paper *The Pythagorean Theorem: I. The finite case* by Kadison, the following subcase of the Schur-Horn Theorem was proved.

**Theorem (Carpenter’s Theorem; 2002)**

Let $A \in \mathcal{M}_n(\mathbb{C})$ be a diagonal, positive contraction such that

$$\text{Tr}(A) = m \in \mathbb{Z}.$$

Then there exists a projection $P \in \mathcal{M}_n(\mathbb{C})$ with $\text{Tr}(P) = m$ such that

$$E_D(P) = A.$$

Kadison also proved a Carpenter’s Theorem for $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ in *The Pythagorean Theorem: II. The infinite discrete case*.
What About Normal Operators?

Definition

A matrix \( a_{i,j} \in M_n(\mathbb{C}) \) is said to be unistochastic if there exists a unitary \( u_{i,j} \in M_n(\mathbb{C}) \) such that

\[
a_{i,j} = |u_{i,j}|^2.
\]

Note

\( \text{diag}(u_{i,j}) \cdot \text{diag}(a_1,...,a_n) = |u_{i,j}|^2 \).

In 1979 Au-Yeung and Pong gave a geometric description of the possible diagonals of a 3 \( \times \) 3 normal matrix.

In 2007 Arverson examined diagonals of normal operators with finite spectrum in \( B(\mathcal{H}) \).

What goes wrong? Let \( T = \text{diag}(0,1,i) \) and \( A = \text{diag}(\frac{1}{2}, i^2, 1+i^2) \).

The matrix

\[
\begin{pmatrix}
\frac{1}{2} & 1 & 0 \\
1 & \frac{1}{2} & 0 \\
0 & 1 & \frac{1}{2}
\end{pmatrix}
\]

is not unistochastic.

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- Note \(\text{diag} ([u_{i,j}]^* \text{diag}(a_1, \ldots, a_n)[u_{i,j}]) = [|u_{i,j}|^2] \cdot (a_1, \ldots, a_n)\).
## Definition

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- **Note:** 
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What About Normal Operators?

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What goes wrong? Let \(T = \text{diag}(0, 1, i)\) and \(A = \text{diag} \left( \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2} \right)\). The matrix

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

is not unistochastic.
Let \((\mathcal{M}, \tau)\) be a type \(\text{II}_1\) factor and let \(T \in \mathcal{M}\) be a normal operator with \(\sigma(T) = \{0, 1, i\}\) and \(\mu_T(\{\lambda\}) = \frac{1}{3}\) for each \(\lambda \in \sigma(T)\).
Let $(\mathcal{M}, \tau)$ be a type II$_1$ factor and let $T \in \mathcal{M}$ be a normal operator with $\sigma(T) = \{0, 1, i\}$ and $\mu_T(\{\lambda\}) = \frac{1}{3}$ for each $\lambda \in \sigma(T)$.

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & i
\end{bmatrix}$$
Let $(\mathcal{M}, \tau)$ be a type II$_1$ factor and let $T \in \mathcal{M}$ be a normal operator with $\sigma(T) = \{0, 1, i\}$ and $\mu_T(\{\lambda\}) = \frac{1}{3}$ for each $\lambda \in \sigma(T)$.

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & i
\end{bmatrix} \sim \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i
\end{bmatrix}
\]
Let \((\mathcal{M}, \tau)\) be a type II\(_1\) factor and let \(T \in \mathcal{M}\) be a normal operator with 
\(\sigma(T) = \{0, 1, i\}\) and \(\mu_T(\{\lambda\}) = \frac{1}{3}\) for each \(\lambda \in \sigma(T)\).

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & i
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & i
\end{bmatrix}
\sim
\begin{bmatrix}
\frac{1}{2} & * & 0 & 0 & 0 & 0 \\
* & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i}{2} & * & 0 & 0 \\
0 & 0 & * & \frac{i}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1+i}{2} & * \\
0 & 0 & 0 & 0 & * & \frac{1+i}{2}
\end{bmatrix}
\]
Theorem (Kennedy, Skoufranis; 2014)

Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{A}$ be a MASA of $\mathcal{M}$, let $E_A : \mathcal{M} \to \mathcal{A}$ be a conditional expectation, and let $\{A_k\}_{k=1}^n \subseteq \mathcal{A}$ be positive operators such that $\sum_{k=1}^n A_k = I_\mathcal{M}$. Then:

1. If $\mathcal{M} = B(\mathcal{H})$ and $\mathcal{A}$ is either a continuous MASA or the diagonal MASA with $E_A$ normal in the case $\mathcal{A} = \mathcal{D}$, then for every $\epsilon > 0$ there exists a collection of pairwise orthogonal projections $\{P_k\}_{k=1}^n \subseteq \mathcal{A}$ such that $\sum_{k=1}^n P_k = I_\mathcal{H}$, $\sigma_e(P_k) = \sigma(P_k)$, and $\|E_A(P_k) - A_k\| < \epsilon$ for all $k \in \{1, \ldots, n\}$.

2. If $\mathcal{M}$ is a type $II_1$ factor with tracial state $\tau$ and $E_A$ is normal, then for every $\epsilon > 0$ there exists a collection of pairwise orthogonal projections $\{P_k\}_{k=1}^n \subseteq \mathcal{A}$ such that $\tau(P_k) = \tau(A_k)$ and $\|E_A(P_k) - A_k\| < \epsilon$ for all $k \in \{1, \ldots, n\}$.

Similar results in type $II_\infty$ and type III factors.
Theorem (Kennedy, Skoufranis; 2014)

Let $\mathcal{A}$ be a MASA in $\mathcal{B}(\mathcal{H})$, let $E_\mathcal{A} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ be a conditional expectation, and let $N \in \mathcal{B}(\mathcal{H})$ be normal.

1. If $\mathcal{A}$ is a continuous MASA, then

$$\{ E_\mathcal{A}(U^*NU) \mid U \in \mathcal{U}(\mathcal{H}) \} = \{ A \in \mathcal{A} \mid \sigma(A) \subseteq \text{conv}(\sigma_e(N)) \}.$$

2. If $\mathcal{A} = \mathcal{D}$, $E_\mathcal{A}$ is normal, and $\sigma(N) \subseteq \text{conv}(\sigma_e(N))$, then

$$\{ E_\mathcal{A}(U^*NU) \mid U \in \mathcal{U}(\mathcal{H}) \} = \{ A \in \mathcal{D} \mid \sigma(A) \subseteq \text{conv}(\sigma_e(N)) \}.$$
Theorem (Kennedy, Skoufranis; 2014)

Let $(\mathcal{M}, \tau)$ be a type II$_1$ factor, let $A$ be a MASA of $\mathcal{M}$, let $E_A : \mathcal{M} \to A$ be the normal conditional expectation of $\mathcal{M}$ onto $A$. Let $N \in \mathcal{M}$ be a normal operator such that $\sigma(N) = \{z_k\}_{k=1}^n \subseteq \mathbb{C}$. Then

$$A \in \{E_A(U^*NU) \mid U \in \mathcal{U}(\mathcal{M})\}$$

if and only if there exists $\{A_k\}_{k=1}^n \subseteq A$ such that

$$0 \leq A_k \leq I_\mathcal{M}, \quad \tau(A_k) = \tau(\chi_{\{z_k\}}(N)), \quad \sum_{k=1}^{n} A_k = I_\mathcal{M},$$

and

$$\sum_{k=1}^{n} z_k A_k = A.$$
Solution to a question posed by Mirsky in 1964:

Theorem (Thompson; 1977 — Sing)

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $a_1, \ldots, a_n \in \mathbb{C}$ be such that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0 \text{ and } |a_1| \geq |a_2| \geq \cdots |a_n| \geq 0.$$

There exists a complex $n$ by $n$ matrix with singular values are $\alpha_1, \ldots, \alpha_n$ and diagonal entries $a_1, \ldots, a_n$ if and only if

1. $\sum_{j=1}^{k} |a_j| \leq \sum_{j=1}^{k} \alpha_j$ for all $k \in \{1, \ldots, n\}$, and
2. $-|a_n| + \sum_{j=1}^{n-1} |a_j| \leq -\alpha_n + \sum_{j=1}^{n-1} \alpha_j$. 

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Definition (Fack; 1982)

Let \((\mathcal{M}, \tau)\) be a type II$_1$ factor, let \(T \in \mathcal{M}\), and let \(t \in [0, 1]\). The \(t^{th}\)-singular number of \(T\) is

\[
\mu_t(T) := \inf \{ \|TP\| \mid P \in \text{Proj}(\mathcal{M}), \tau(I_{\mathcal{M}} - P) \leq t \}.
\]
Singular Values in $\text{II}_1$ Factors

**Definition (Fack; 1982)**

Let $(\mathcal{M}, \tau)$ be a type $\text{II}_1$ factor, let $T \in \mathcal{M}$, and let $t \in [0, 1]$. The $t^{\text{th}}$-singular number of $T$ is

$$
\mu_t(T) := \inf \{ \| TP \| \mid P \in \text{Proj}(\mathcal{M}), \tau(I_{\mathcal{M}} - P) \leq t \}.
$$

Furthermore

$$
\{ UTV \mid U, V \in \mathcal{U}(\mathcal{M}) \} = \{ R \in \mathcal{M} \mid \mu_t(R) = \mu_t(T) \text{ for all } t \in [0, 1] \}. 
$$
Definition

Let \((\mathcal{M}, \tau)\) be a type II\(_1\) factor. For two operators \(A, S \in \mathcal{M}\) we say that \(S\) submajorizes \(A\), denoted \(A \prec_w S\), if

\[
\int_0^t \mu_s(A) \, ds \leq \int_0^t \mu_s(S) \, ds
\]

for all \(t \in [0, 1]\).

Note, if \(A\) and \(S\) are positive, then

\[
\tau(A) = \tau(S) \quad \text{and} \quad A \prec_w S \iff A \prec S.
\]
Let $(\mathcal{M}, \tau)$ be a type II$_1$ factor, let $\mathcal{A}$ be a MASA of $\mathcal{M}$, and $E_\mathcal{A} : \mathcal{M} \to \mathcal{A}$ be the normal conditional expectation.

**Theorem (Kennedy, Skoufranis; 2014)**

If $T \in \mathcal{M}$, then $E_\mathcal{A}(T) \prec_w T$.

**Question**

For $A \in \mathcal{A}$ and $T \in \mathcal{M}$ with $A \prec_w T$, does there exists an $S \in \{UTV \mid U, V \in U(\mathcal{M})\}$ such that $E_\mathcal{A}(S) = A$?
Let \((\mathcal{M}, \tau)\) be a type \(\text{II}_1\) factor, let \(\mathcal{A}\) be a MASA of \(\mathcal{M}\), and \(E_\mathcal{A} : \mathcal{M} \rightarrow \mathcal{A}\) be the normal conditional expectation.

**Theorem (Kennedy, Skoufranis; 2014)**

If \(T \in \mathcal{M}\) and \(A \in \mathcal{A}\) be such that \(A \prec_w T\), then for every \(\epsilon > 0\) there exists unitary operators \(U, V \in \mathcal{M}\) such that

\[
\|E_\mathcal{A}(UTV) - A\| < \epsilon.
\]
Theorem (Kennedy, Skoufranis; 2014)
Let \((\mathcal{M}, \tau)\) be a type II$_1$ factor, let \(\mathcal{A}\) be a MASA of \(\mathcal{M}\), and \(E_{\mathcal{A}} : \mathcal{M} \to \mathcal{A}\) be the normal conditional expectation. The following are equivalent:

1. If \(T \in \mathcal{M}\) and \(A \in \mathcal{A}\) are self-adjoint and \(A \prec T\), then there exists an \(S \in \mathcal{M}\) such that \(T\) and \(S\) are approximately unitarily equivalent and
   \[E_{\mathcal{A}}(S) = A.\]

2. If \(T \in \mathcal{M}\) and \(A \in \mathcal{A}\) are such that \(A \prec_w T\), then there exists an \(S \in \mathcal{M}\) such that \(T\) and \(S\) have the same singular values and
   \[E_{\mathcal{A}}(S) = A.\]
Thanks for Listening!