Diagonals of Certain Operators in von Neumann Algebras

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Diagonals in a Von Neumann Algebras

Diagonal of a Matrix

Let $\mathcal{D}_n$ be the diagonal subalgebra of $\mathcal{M}_n(\mathbb{C})$ and let $E_{\mathcal{D}_n} : \mathcal{M}_n(\mathbb{C}) \to \mathcal{D}_n$ be defined by

$$E_{\mathcal{D}_n}([a_{i,j}]) = (a_{1,1}, a_{2,2}, \ldots, a_{n,n}).$$

Given $T \in \mathcal{M}_n(\mathbb{C})$ with fixed properties, what values can $E_{\mathcal{D}_n}(T)$ take?

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Analogue in von Neumann Algebras

Let $\mathfrak{M}$ be a von Neumann algebra, let $\mathcal{A}$ be a MASA in $\mathfrak{M}$, and let

$$E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$$

be a conditional expectation of $\mathfrak{M}$ onto $\mathcal{A}$. Given $T \in \mathfrak{M}$ with fixed properties, what operators may $E_{\mathcal{A}}(T)$ be?

The Schur-Horn Theorem

Theorem (Schur; 1923)

If $T \in \mathcal{M}_n(\mathbb{C})$ is a self-adjoint matrix with eigenvalues and diagonal entries

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \qquad a_1 \geq a_2 \geq \cdots \geq a_n$$

respectively, then

1. $\sum_{k=1}^m a_k \leq \sum_{k=1}^m \lambda_k$ *for all* $m \in \{1, \ldots, n\}$, *and*
2. $\sum_{k=1}^n a_k = \sum_{k=1}^n \lambda_k$.

Theorem (Horn; 1954)

If

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \qquad a_1 \geq a_2 \geq \cdots \geq a_n$$

are elements of $\mathbb{R}$ such that the above two conditions hold, then there exists a self-adjoint matrix $T \in \mathcal{M}_n(\mathbb{C})$ with eigenvalues $(\lambda_k)_{k=1}^n$ and diagonal entries $(a_k)_{k=1}^n$.

The Carpenter's Theorem

In the beautiful paper *The Pythagorean Theorem: I. The finite case* by Kadison, the following subcase of the Schur-Horn Theorem was proved.

Theorem (Carpenter's Theorem; 2002)

Let $A \in \mathcal{M}_n(\mathbb{C})$ *be a diagonal, positive contraction such that*

$$Tr(A) = m \in \mathbb{Z}.$$

Then there exists a projection $P \in \mathcal{M}_n(\mathbb{C})$ *with* $Tr(P) = m$ *such that*

$$E_{\mathcal{D}}(P) = A.$$

Kadison also proved a Carpenter's Theorem for $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ in *The Pythagorean Theorem: II. The infinite discrete case*.

What About Normal Operators?

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Definition

A matrix $[a_{i,j}] \in \mathcal{M}_n(\mathbb{C})$ is said to be unistochastic if there exists a unitary $[u_{i,j}] \in \mathcal{M}_n(\mathbb{C})$ such that $a_{i,j} = |u_{i,j}|^2$.

- Note $\mathrm{diag}\left([u_{i,j}]^* \mathrm{diag}(a_1, \ldots, a_n)[u_{i,j}]\right) = [|u_{i,j}|^2] \cdot (a_1, \ldots, a_n)$.

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What goes wrong? Let $T = \mathrm{diag}(0, 1, i)$ and $A = \mathrm{diag}\left(\frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}\right)$. The matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is not unistochastic.

What Goes Right for Normals in II_1 Factors?

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor and let $T \in \mathfrak{M}$ be a normal operator with $\sigma(T) = \{0, 1, i\}$ and $\mu_T(\{\lambda\}) = \frac{1}{3}$ for each $\lambda \in \sigma(T)$.

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Approximate Mulit-Variate Carpenter's Theorem

Theorem (Kennedy, Skoufranis; 2014)

Let $\mathfrak{M}$ be a von Neumann algebra, let $\mathcal{A}$ be a MASA of $\mathfrak{M}$, let $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ be a conditional expectation, and let $\{A_k\}_{k=1}^n \subseteq \mathcal{A}$ be positive operators such that $\sum_{k=1}^n A_k = I_{\mathfrak{M}}$. Then:

1. *If $\mathfrak{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{A}$ is either a continuous MASA or the diagonal MASA with $E_{\mathcal{A}}$ normal in the case $\mathcal{A} = \mathcal{D}$, then for every $\epsilon > 0$ there exists a collection of pairwise orthogonal projections $\{P_k\}_{k=1}^n \subseteq \mathcal{A}$ such that $\sum_{k=1}^n P_k = I_{\mathcal{H}}$, $\sigma_e(P_k) = \sigma(P_k)$, and $\|E_{\mathcal{A}}(P_k) - A_k\| < \epsilon$ for all $k \in \{1, \ldots, n\}$.*
2. *If $\mathfrak{M}$ is a type II_1 factor with tracial state τ and $E_{\mathcal{A}}$ is normal, then for every $\epsilon > 0$ there exists a collection of pairwise orthogonal projections $\{P_k\}_{k=1}^n \subseteq \mathcal{A}$ such that $\tau(P_k) = \tau(A_k)$ and $\|E_{\mathcal{A}}(P_k) - A_k\| < \epsilon$ for all $k \in \{1, \ldots, n\}$.*

Similar results in type II_∞ and type III factors.

Approximate Schur-Horn for Normals in $\mathcal{B}(\mathcal{H})$

Theorem (Kennedy, Skoufranis; 2014)

Let $\mathcal{A}$ be a MASA in $\mathcal{B}(\mathcal{H})$, let $E_{\mathcal{A}} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ be a conditional expectation, and let $N \in \mathcal{B}(\mathcal{H})$ be normal.

1. *If $\mathcal{A}$ is a continuous MASA, then*
$$\overline{\{E_{\mathcal{A}}(U^*NU) \mid U \in \mathcal{U}(\mathcal{H})\}} = \{A \in \mathcal{A} \mid \sigma(A) \subseteq \mathrm{conv}(\sigma_e(N))\}.$$
2. *If $\mathcal{A} = \mathcal{D}$, $E_{\mathcal{A}}$ is normal, and $\sigma(N) \subseteq \mathrm{conv}(\sigma_e(N))$, then*
$$\overline{\{E_{\mathcal{A}}(U^*NU) \mid U \in \mathcal{U}(\mathcal{H})\}} = \{A \in \mathcal{D} \mid \sigma(A) \subseteq \mathrm{conv}(\sigma_e(N))\}.$$

Approximate Schur-Horn for Normals in II_1 Factors

Theorem (Kennedy, Skoufranis; 2014)

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $\mathcal{A}$ be a MASA of $\mathfrak{M}$, let $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ be the normal conditional expectation of $\mathfrak{M}$ onto $\mathcal{A}$. Let $N \in \mathfrak{M}$ be a normal operator such that $\sigma(N) = \{z_k\}_{k=1}^n \subseteq \mathbb{C}$. Then

$$A \in \overline{\{E_{\mathcal{A}}(U^* N U) \mid U \in \mathcal{U}(\mathfrak{M})\}}$$

if and only if there exists $\{A_k\}_{k=1}^n \subseteq \mathcal{A}$ such that

$$0 \leq A_k \leq I_{\mathfrak{M}}, \qquad \tau(A_k) = \tau(\chi_{\{z_k\}}(N)), \qquad \sum_{k=1}^n A_k = I_{\mathfrak{M}},$$

and

$$\sum_{k=1}^n z_k A_k = A.$$

Diagonals Based on Prescribed Singular Values

Solution to a question posed by Mirsky in 1964:

Theorem (Thompson; 1977 — Sing)

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ *and* $a_1, \ldots, a_n \in \mathbb{C}$ *be such that*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0 \text{ and } |a_1| \geq |a_2| \geq \cdots |a_n| \geq 0.$$

There exists a complex n *by* n *matrix with singular values are* $\alpha_1, \ldots, \alpha_n$ *and diagonal entries* $a_1, \ldots, a_n$ *if and only if*

1. $\sum_{j=1}^{k} |a_j| \leq \sum_{j=1}^{k} \alpha_j$ *for all* $k \in \{1, \ldots, n\}$*, and*
2. $-|a_n| + \sum_{j=1}^{n-1} |a_j| \leq -\alpha_n + \sum_{j=1}^{n-1} \alpha_j$.

Singular Values in II_1 Factors

Definition (Fack; 1982)

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $T \in \mathfrak{M}$, and let $t \in [0, 1]$. The t^{th}-singular number of T is

$$\mu_t(T) := \inf\{\|TP\| \mid P \in \mathrm{Proj}(\mathfrak{M}), \tau(I_{\mathfrak{M}} - P) \leq t\}.$$

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Furthermore

$$\overline{\{UTV \mid U, V \in \mathcal{U}(\mathfrak{M})\}} = \{R \in \mathfrak{M} \mid \mu_t(R) = \mu_t(T) \text{ for all } t \in [0, 1]\}.$$

Submajorization Based on Singular Values

Definition

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor. For two operators $A, S \in \mathfrak{M}$ we say that S submajorizes A, denoted $A \prec_w S$, if

$$\int_0^t \mu_s(A)\, ds \leq \int_0^t \mu_s(S)\, ds$$

for all $t \in [0, 1]$.

Note, if A and S are positive, then

$$\tau(A) = \tau(S) \text{ and } A \prec_w S \iff A \prec S.$$

Singular Values of Expectations

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $\mathcal{A}$ be a MASA of $\mathfrak{M}$, and $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ be the normal conditional expectation.

Theorem (Kennedy, Skoufranis; 2014)

If $T \in \mathfrak{M}$, then $E_{\mathcal{A}}(T) \prec_w T$.

Question

For $A \in \mathcal{A}$ and $T \in \mathfrak{M}$ with $A \prec_w T$, does there exists an

$$S \in \overline{\{UTV \mid U, V \in \mathcal{U}(\mathfrak{M})\}}$$

such that $E_{\mathcal{A}}(S) = A$?

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $\mathcal{A}$ be a MASA of $\mathfrak{M}$, and $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ be the normal conditional expectation.

Theorem (Kennedy, Skoufranis; 2014)

If $T \in \mathfrak{M}$ and $A \in \mathcal{A}$ be such that $A \prec_w T$, then for every $\epsilon > 0$ there exists unitary operators $U, V \in \mathfrak{M}$ such that

$$\|E_{\mathcal{A}}(UTV) - A\| < \epsilon.$$

Equivalence with the Schur-Horn Theorem in II_1 Factors

Theorem (Kennedy, Skoufranis; 2014)

Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $\mathcal{A}$ be a MASA of $\mathfrak{M}$, and $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ be the normal conditional expectation. The following are equivalent:

1. *If $T \in \mathfrak{M}$ and $A \in \mathcal{A}$ are self-adjoint and $A \prec T$, then there exists an $S \in \mathfrak{M}$ such that T and S are approximately unitarily equivalent and*
$$E_{\mathcal{A}}(S) = A.$$
2. *If $T \in \mathfrak{M}$ and $A \in \mathcal{A}$ are such that $A \prec_w T$, then there exists an $S \in \mathfrak{M}$ such that T and S have the same singular values and*
$$E_{\mathcal{A}}(S) = A.$$

Thanks for Listening!