Rank Constrained Homotopies

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42nd Canadian Operator Algebras Symposium
June 23rd, 2014
Notation and Statement of the Main Theorem

Notation

- $X$: Compact Hausdorff space of finite covering dimension.
- $(M_n)_+$: non-negative definite $n \times n$ matrices over $\mathbb{C}$
- $S(n, k, l) = \{ b \in (M_n)_+ | l \leq \text{rank}(b) \leq k \}$
- $F(X, S(n, k, l)) = \{ f: X \to S(n, k, l) : f \text{ is continuous} \}$

Theorem

For any $n, k, l \in \mathbb{N}$, if $\left\lfloor \frac{\text{dim } X}{2} \right\rfloor \leq k - l$, then $F(X, S(n, k, l))$ is path connected. In particular, $\forall n, k, l \in \mathbb{N}, \pi_r(S(n, k, l)) = 0$, whenever $r \leq 2(k - l) + 1$. 
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Motivation

- Homotopy properties of the space $S(n, k, l)$ has applications in $C^*$-algebra theory.
- Let $A$ be a unital $C^*$-algebra with $T(A) \neq \emptyset$, where $T(A)$ is the tracial state space of $A$.
- Any $a \in A_+$, induce a lower semi-continuous affine function $\iota_a$ on $T(A)$, given by,

$$
\iota_a(\tau) = \lim_{n} (\tau(a^{1/n}))
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- If $K$ denote the compacts on a separable Hilbert space, $\iota_a$ extends to $(A \otimes K)_+$ in a natural way.
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Question?
For a unital, simple $C^*$-algebra $A$, is it possible to approximate strictly positive continuous affine functions $(\text{Aff}(T(A))_{++})$ by functions of the form $\iota_a, a \in (A \otimes K)_+$?

(Andrew Toms, 2009), For (unital, simple) ASH algebras with slow dimension growth, the question has a positive answer.

Following is a key proposition in the proof of the above.

Lemma (Toms)

For any $n, k, l \in \mathbb{N}$, if $4\dim X \leq k - l$, then $F(X, S(n, k, l))$ is path connected.
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Recap from Vector bundle theory

Recall...

- A (complex) vector bundle over $X$ is a triple $(E, p, X)$, where $E$ is a topological space, $p: E \rightarrow X$ is a continuous map with each fiber $p^{-1}(x) = E_x$ admitting a $\mathbb{C}$-vector space structure.
- Let $\theta^k(X) = (X \times \mathbb{C}^k, \pi, X)$, where $\pi$ is the projection onto $X$. $\theta^k$ is called the $k$-dimensional product bundle.
- $\alpha = (E, p, X)$ is called trivial if $\alpha \cong \theta^k(X)$ for some $k$.

Definition

$\alpha = (E, p, X)$ is locally trivial if $X$ has an open covering $\{U_{\lambda}\}$ such that $\alpha|_{U_{\lambda}} \cong \theta^k(U_{\lambda})$ for each $\lambda$. 
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Rank Constrained Homotopies
Stability properties of locally trivial bundles

Let $\text{Bun}_k(X)$ denote the category of all locally trivial (Complex) vector bundles over $X$, of dimension $k$.

**Theorem**

Let $X$ be a (para) compact, Hausdorff and finite dimensional topological space.

1. If $\alpha \in \text{Bun}_k(X)$, then there is a trivial vector bundle $\delta$ over $X$ with $\dim \delta \geq k - \left\lfloor \frac{\dim X}{2} \right\rfloor$, such that $\delta$ is a direct summand of $\alpha$. i.e. $\alpha = \delta \oplus \eta$, for some bundle $\eta$.

2. Let $\alpha, \beta \in \text{Bun}_k(X)$. If $k \geq \frac{\dim X}{2}$ and $\alpha \oplus \gamma \sim \beta \oplus \gamma$ for some bundle $\gamma$ over $X$, then $\alpha \sim \beta$. 
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Given a projection $p \in M_n(C(X))$, there is an associated vector bundle $\epsilon(p) = (E_p, \pi, X)$, with

$$E(p) = \{(x, v) : x \in X, v \in p(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n.$$  

Moreover, every locally trivial vector bundle over $X$ can be realized in the above form [R. G. Swan, 1961]. That is, fixed $n \geq k + \frac{\dim X}{2}$, for each $\alpha \in \text{Bun}_k(X)$ there is a projection $p_\alpha \in M_n(C(X))$ such that $\epsilon(p_\alpha) \cong \alpha$. Moreover, $\alpha \cong \beta$ iff $p_\alpha \sim_{M.V} p_\beta$. 
Projections and Vector Bundles

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Outline of the proof

Given \( a, b \in F(X, S(n, k, l)) \) with \( k - l \geq \left\lfloor \frac{\dim X}{2} \right\rfloor \), need to show that \( \exists h: [0, 1] \rightarrow F(X, (n, k, l)) \) with \( h(0) = a, h(1) = b \).

**Lemma (A)**

If \( k - l \geq \left\lfloor \frac{\dim X}{2} \right\rfloor \), then \( \forall a \in F(X, S(n, k, l)), \exists p \in M_n(C(X)) \) a trivial projection of rank \( l \) such that

\[
\dim[(p(x) + a(x))(\mathbb{C}^n)] \leq k, \forall x \in X
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**Lemma (B)**

Suppose \( n \geq l + \frac{\dim X}{2} \) and let \( p, q \in M_n(C(X)) \) be trivial projections of rank \( l \). Then \( \exists h: [0, 1] \rightarrow \text{Proj}(M_n(C(X))) \) with \( h(0) = p \) and \( h(1) = q \).
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Idea of the proof of Lemma A

- Given \( a \in F(X, S(n, k, l)) \) there associates a vector bundle \( \epsilon(a) = (E(a), p, X) \). Here,

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  and \( p \) is the restriction of canonical projection of \( X \times \mathbb{C}^n \) onto \( X \), to \( E(a) \). The bundle \( \epsilon(a) \) is not necessarily locally trivial.

- However, we can partition \( X \) so that the restriction \( \epsilon(a) \) to each set in the partition is a locally trivial bundle.

- Then, to get the trivial projection given in the conclusion of Lemma A, we apply stability properties of locally trivial bundles discussed before, to restricted bundles.

- To define the trivial projection globally we will use some extension results due to Chris Phillips and Andrew Toms.
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Let $a \in F(X, S(n, k, l))$ and suppose the rank values of $a$ are $n_1 < n_2 < \ldots < n_L$. For simplicity let $\epsilon = \epsilon(a)$.

For $1 \leq i \leq L$, set $E_i = \{x \in X : \text{rank } a(x) = n_i\}$. Then $\epsilon|_{E_i}$ is locally trivial. The support projection of $a|_{E_i}$ is continuous and $\epsilon|_{E_i}$ is the bundle corresponding to this projection.

**Definition (Toms)**

A positive element $a \in M_n(C(X))_+$ is well-supported, if
\[ \forall 1 \leq i \leq L, \exists p_i : E_i \to \text{Proj}(M_n) \text{ such that for each } i, \]
\[ p_i(x) = \lim_{n \to \infty} (a(x))^{1/n}, \forall x \in E_i \text{ and } \]
for each pair $(i, l)$ with $j \geq i$,
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\]

for each pair \((i, l)\) with \( j \geq i \),

\[
p_j(x) \leq p_i(x), \quad \forall x \in \overline{E_i} \cap \overline{E_j}
\]
Idea of the proof of Lemma A

**Theorem (Toms)**

\[\text{Let } a \in M_n(C(X))_+ \text{ then there exists a well-supported } b \in M_n(C(X))_+ \text{ such that the set of rank values of } b \text{ is same as the set of rank values of } a \text{ and } b \leq a.\]

**Corollary**

\[\text{Given } a \in F(X, S(n, k, l)), \text{ there is } b \in F(X, S(n, k, l)) \text{ with } b \text{ homotopic to } a.\]

- Corollary reduces the proof of Lemma A to the case of \(a \in F(X, S(n, k, l))\) being well-supported.
**Introduction and Motivation**

**Classical Vector Bundle Theory**

**Proof of the main Theorem**

Continuity of path connectedness of \( F(X, S(n, k, l)) \)

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**Idea of the proof of Lemma A**

**Theorem (Toms)**

Let \( a \in M_n(C(X))_+ \) then there exists a well-supported \( b \in M_n(C(X))_+ \) such that the set of rank values of \( b \) is same as the set of rank values of \( a \) and \( b \leq a \).

**Corollary**

Given \( a \in F(X, S(n, k, l)) \), there is \( b \in F(X, S(n, k, l)) \) with \( b \) homotopic to \( a \).

Corollary reduces the proof of Lemma A to the case of \( a \in F(X, S(n, k, l)) \) being well-supported.

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Idea of the proof of Lemma A

**Theorem (Toms)**

Let $a \in M_n(C(X))_+$ then there exists a well-supported $b \in M_n(C(X))_+$ such that the set of rank values of $b$ is same as the set of rank values of $a$ and $b \preceq a$.

**Corollary**

Given $a \in F(X, S(n, k, l))$, there is $b \in F(X, S(n, k, l))$ with $b$ homotopic to $a$.

Corollary reduces the proof of Lemma A to the case of $a \in F(X, S(n, k, l))$ being well-supported.
Proof of Lemma A

Let \( a \in F(X, S(n, k, l)) \) be well supported and choose \( E_1, E_2, \ldots, E_L \) and \( p_1, p_2, \ldots, p_L \) be as given by the definition of well-supportedness. Write \( F_i = \overline{E_i} \).

Choose a trivial projection \( q_1 \in M_n(C(F_1)) \), with

\[
\text{rank } q_1 = n_1 - \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad q_1 \leq p_1.
\]

By Corollaries 1 and 2, extend \( q \) to a trivial projection \( q_1 \in M_n(C(X)) \) such that \( \forall 1 \leq i \leq L, \)

\[
q_1(x) \leq p_i(x), \forall x \in F_i.
\]

Since \( q_1 \) is trivial and \( n - \text{rank } q_1 \geq k - l \geq \left\lfloor \frac{d}{2} \right\rfloor \),

\[
q_1^\perp = 1_n - q_1 \quad \text{is also trivial and hence}
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q_1^\perp M_n(C(X))q_1^\perp \cong M_{n-n_1+\left\lfloor \frac{d}{2} \right\rfloor}(C(X))
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Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank $l$. Then $\exists h: [0, 1] \rightarrow \text{Proj}(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

Sketch of the proof of Lemma B:
- Lemma holds when $X$ is a $CW$-complex. (Is a consequence of the homotopy classification theorem for bundles over $CW$-complexes)
- If $X$ is a compact metric space, $X$ is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha, \beta}\}$ of finite $CW$-complexes and we use a standard approximation argument to prove the result.
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Theorem

Suppose for each finite simplicial complex $Z$ with $\dim Z \leq d$, the function space $F(Z, S(n, k, l))$ is path connected. Then $F(X, S(n, k, l))$ is path connected for any compact Hausdorff space $X$ with $\dim X \leq d$.

Corollary

If $\pi_r(S(n, k, l)) = 0$ for each $r \leq d$ then, $F(X, S(n, k, l))$ is path connected for any compact Hausdorff space $X$ with $\dim X \leq d$.

Fixed $n, k, l \in \mathbb{N}$ with $n > k > l$ find a non vanishing homotopy group of $S(n, k, l)$?
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Thank you!
Some required definitions and results...

**Theorem (Chris Phillips)**

Let $X$ be a compact, Hausdorff space of dimension $d$, and let $Y \subset X$ be closed. Let $p, q \in M_n(C(X))$ be projections with the property that $\text{rank}(q(x)) + \left\lfloor \frac{d}{2} \right\rfloor \leq \text{rank}(p(x))$, $\forall x \in X$. Let $s_0 \in M_n(C(Y))$ be such that $s_0^* s_0 = q|_Y$ and $s_0 s_0^* \leq p|_Y$. It follows that there is $s \in M_n(C(X))$ such that $s^* s = q$, $ss^* \leq p$, and $s_0 = s|_Y$.

**Corollary (1)**

Any trivial projection $q \in M_n(C(Y))$ with $\text{rank}(q) \leq n - \left\lfloor \frac{d}{2} \right\rfloor$, extends to a trivial projection in $M_n(C(X))$. 
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Corollary (2, Andrew Toms)

Let $q \in M_n(C(X))$ and $F_1, ..., F_k$ be a closed cover of $X$.

\( \forall 1 \leq i \leq k, \) let $p_i \in \text{Proj}(M_n(C(F_i)))$ of constant rank $n_i$.

Assume $n_1 < n_2 < \cdots < n_k$ and $p_i(x) \leq p_j(x), \forall i \leq j, x \in F_i \cap F_j$.

Say $n_i - \text{rank } (q) \geq \left\lfloor \frac{d}{2} \right\rfloor, \forall 1 \leq i \leq L$. The following hold.

If $Y \subset X$ is closed, $q \upharpoonright_Y$ is trivial and,

\[
q(y) \leq \bigwedge_{\{i | y \in E_i\}} p_i(y), \forall y \in Y,
\]

then $q \upharpoonright_Y$ extends to trivial a projection $\tilde{q}$ on $X$ with,

\[
\tilde{q}(x) \leq \bigwedge_{\{i | x \in E_i\}} p_i(x), \forall x \in X.
\]
Proof of Lemma A

For each $1 \leq i \leq L$, let $p_i^{(1)}(x) = p_i(x) - q_1(x)$, $\forall x \in F_i$. Then,

$$p_i^{(1)} \in q_1 \perp M_n(C(F_i)) q_1 \perp \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(F_i))$$

Choose a trivial projection $q_2 \in M_n(F_2)$ with $q_2 \leq p_2^{(1)}$ and

$$\text{rank } q_2 = n_2 - (n_1 - \left\lfloor \frac{d}{2} \right\rfloor) - \left\lfloor \frac{d}{2} \right\rfloor = n_2 - n_1.$$

Write $X_1 = F_2 \cup F_2 \cup \ldots \cup F_L$. Extend $q_2$ to a trivial projection $q_2 \in M_n(C(X_1))$, such that $\forall 2 \leq i \leq L$,

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Extend $q_2$ to a trivial projection $q_2 \in M_n(C(X))$. 
Proof of Lemma A

For each $1 \leq i \leq L$, let $p_i^{(1)}(x) = p_i(x) - q_1(x), \forall x \in F_i$. Then,

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Proof of Lemma A

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- Extend $q_2$ to a trivial projection $q_2 \in M_n(C(X))$. 

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Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\text{rank}(a(x) + Q_2(x)) \leq \text{rank} p_1 + \text{rank} q_2$$

$$= n_1 + (n_2 - n_1) = n_2$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,

$$\text{rank}(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$$

$$\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k, \forall x \in X.$$
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- $\text{rank}(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
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Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.

For any $x \in F_1$,

$$\text{rank}(a(x) + Q_2(x)) \leq \text{rank } p_1 + \text{rank } q_2$$

$$= n_1 + (n_2 - n_1) = n_2$$

Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,

- $\text{rank } (Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
- $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k, \forall x \in X$. 
Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with
  \[
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- If \( n_L = k \), rank \( Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l \). Setting \( p = Q_L \) completes the proof of the Lemma.
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Now \( p = Q_L + r \) is trivial with rank \( p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l \) and,

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This completes the proof of Lemma A.
Homotopy Classification of Vector bundles

- \( V_k(\mathbb{C}^n) = \{(v_1, v_2, ... v_k) : v_i \in \mathbb{C}^n \text{ and } <v_i|v_j> = \delta_{i,j}\} \).
- The complex Grassmann variety, \( G_k(\mathbb{C}^n) \), is given by
  \[
  G_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim,
  \]
  where \((v_1, v_2, ..v_k) \sim (w_1, w_2, ...w_k)\) iff the two \( k \)-tuples span the same subspace in \( \mathbb{C}^n \).
- We have the natural inclusion, \( G_k(\mathbb{C}^n) \subset G_k(\mathbb{C}^{n+1}) \), induced by \( \mathbb{C}^n \subset \mathbb{C}^{n+1} \). Set \( G_k(\mathbb{C}^\infty) = \bigcup_{n\in\mathbb{N}} G_k(\mathbb{C}^n) \).
- Let \( \gamma^n_k = (E, \pi, G_k(\mathbb{C}^n)) \), where
  \[
  E = \{(V, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n : v \in V\},
  \]
  and \( \pi \) is the canonical projection. Then, \( \gamma^n_k \) is a locally trivial vector bundle over \( G_k(\mathbb{C}^n) \). Again, \( \gamma^n_k \subset \gamma^{n+1}_k \). The resulting direct limit, \( \gamma_k = \bigcup_{n\in\mathbb{N}} \gamma^n_k \) determine a vector bundle over \( G_k(\mathbb{C}^\infty) \) of dimension \( k \).
Homotopy Classification of Vector bundles

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  $$G_k(\mathbb{C}^n) = V_k(\mathbb{C}^n) / \sim,$$

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Homotopy Classification of Vector bundles

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Homotopy Classification of Vector bundles

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Homotopy Classification of Vector bundles

- Let \( f : Y \to X \) be a continuous and \( \alpha = (E, p, X) \)
- Let \( f^* (\alpha) = (E(f^* (\alpha)), \pi, Y) \)
  \[ E(f^* (\alpha)) = \{(w, y) \in E \times Y : f(y) = p(w)\} \]
  and \( \pi = \) restriction of the canonical projection.
- \( f^* (\alpha) \) is called the pullback of \( \alpha \) to \( Y \) via \( f \).

**Theorem (Homotopy Classification of Vector bundles)**

The function that maps each homotopy class \([f] : X \to G_k(\mathbb{C}^\infty)\) to the isomorphism class of \( f^* (\gamma_k) \), is a bijection.

- If \( X \) is a finite \( CW \)-complex, then \( G_k(\mathbb{C}^\infty), \gamma_k \) of the above theorem can be replaced by \( G_k(\mathbb{C}^n), \gamma^k \), provided \( n \geq k + \frac{\text{dim} X}{2} \).
Homotopy Classification of Vector bundles

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Theorem (Homotopy Classification of Vector bundles)

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The function that maps each homotopy class $[f] : X \to G_k(\mathbb{C}^\infty)$ to the isomorphism class of $f^*(\gamma_k)$, is a bijection.

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Proof of Lemma B

- Union of all $\psi^T_\alpha(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank $l$, choose $\alpha$ and a projection $p_\alpha \in M_n(C(X))$ with $\|p - \psi^T_\alpha(p_\alpha)\| < 1$.
- $p$ is homotopic to $\psi^T_\alpha(p_\alpha)$. Hence $\psi^T_\alpha(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi^T_\alpha(p_\alpha)$ and $q = \psi^T_\alpha(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi^T_\alpha(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As $Y_\alpha$ is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$. 
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- Union of all $\psi^T_\alpha(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank $l$, choose $\alpha$ and a projection $p_\alpha \in M_n(C(X))$ with $\|p - \psi^T_\alpha(p_\alpha)\| < 1$.
- $p$ is homotopic to $\psi^T_\alpha(p_\alpha)$. Hence $\psi^T_\alpha(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi^T_\alpha(p_\alpha)$ and $q = \psi^T_\alpha(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi^T_\alpha(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As $Y_\alpha$ is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

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- May assume, \( p = \psi^T(\tilde{p}_\alpha) \), \( q = \psi^T(\tilde{q}_\alpha) \), where \( \tilde{p}_\alpha, (\tilde{q}_\alpha) \in M_n(C(X_\alpha)) \) are trivial projections of rank \( l \).
- By the homotopy classification of locally trivial vector bundles over \( CW \)-complexes, there is a projection valued path \( H \) in \( M_n(C(X_\alpha)) \) connecting \( \tilde{p}_\alpha \) and \( \tilde{q}_\alpha \).
- Taking \( h = H \circ \psi_\alpha \) completes the proof for compact metric spaces.
- For a general compact Hausdorff space \( X \), \( X \) is the inverse limit of compact metric spaces each of dimension at most the dimension of \( X \). The conclusion now follows from an exact similar argument as before.
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