C*-algebras of Matricially Ordered *-Semigroups

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Universal C*-algebras involving an automorphism realized via an implementing unitary, or an endomorphism via an isometry, have played a fundamental role in operator algebras. Such maps preserve algebraic structure.

A map of a C*-algebra defined via an implementing partial isometry does not preserve algebra structure. It is, however, a completely positive *-linear map.

We consider *-semigroups $S$, matricial partial order orders on $S$, along with a universal C*-algebra associated with $S$ and a matricial ordering on $S$. 
For a particular example of a matrically ordered *-semigroup $S$ along with complete order map on $S$, we obtain a C*-correspondence over the associated C*-algebra of $S$. The complete order map is implemented by a partial isometry in the Cuntz-Pimsner C*-algebra associated with the correspondence.
For a particular example of a matrically ordered $*$-semigroup $S$ along with complete order map on $S$, we obtain a $C^*$-correspondence over the associated $C^*$-algebra of $S$. The complete order map is implemented by a partial isometry in the Cuntz-Pimsner $C^*$-algebra associated with the correspondence.

The resulting Cuntz-Pimsner $C^*$-algebra for this example is the universal $C^*$-algebra $P$ generated by a partial isometry.
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The resulting Cuntz-Pimsner C*-algebra for this example is the universal C*-algebra $\mathcal{P}$ generated by a partial isometry.

It is known that $\mathcal{P}$ is nonunital, nonexact, residually finite dimensional, and Morita equivalent to the universal C*-algebra generated by a contraction.
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For $B$ a C*-algebra, the contractions (or strict contractions) in $B$ viewed as a semigroup under multiplication, with * the usual involution. In particular, for $\mathcal{H}$ a Hilbert space and $B = B(\mathcal{H})$. 
Matricial order

For a semigroup $S$ the set of $k \times k$ matrices with entries in $S$, $M_k(S)$, does not inherit much algebraic structure through $S$. However, the $*$-structure, along with multiplication of specific types of matrices over $S$ is sufficient to provide some context for an order structure.
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For $k \in \mathbb{N}$, let $[n_i]$ denote an element $[n_1, ..., n_k] \in M_{1,k}(S)$, the $1 \times n$ matrices with entries in $S$.

Then $[n_i]^* \in M_{k,1}(S)$, a $k \times 1$ matrix over $S$, 

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Then $[n_i]^* \in M_{k,1}(S)$, a $k \times 1$ matrix over $S$, and the element $[n_i]^*[n_j] = [n_i^*n_j] \in M_k(S)^{sa}$. 
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For example, if

$$
\begin{pmatrix}
  a_{1,1} & a_{1,2} \\
  a_{2,1} & a_{2,2}
\end{pmatrix}
$$

is positive in $M_2(B)^{sa}$ then

$$
\begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,2} \\
  a_{2,1} & a_{2,2} & a_{2,2} \\
  a_{2,1} & a_{2,2} & a_{2,2}
\end{pmatrix}
$$

is also positive in $M_3(B)^{sa}$. 
We may describe this property using $d$-tuples of natural numbers as ordered partitions of $k$ where zero summands are allowed.
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Notation:
For $d, k \in \mathbb{N}$ and $d \leq k$, set

$$\mathcal{P}(d, k) = \left\{ (t_1, \ldots, t_d) \in (\mathbb{N}_0)^d \mid \sum_{r=1}^{d} t_r = k \right\}.$$
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\]

Each \(\tau = (t_1, \ldots, t_d) \in \mathcal{P}(d, k)\) yields a *-map \(\iota_\tau : M_d(B) \to M_k(B)\). For \([a_{i,j}] \in M_d(B)\) the element \(\iota_\tau([a_{i,j}]) := [a_{i,j}]_\tau \in M_k(B)\) is the matrix obtained using matrix blocks; the \(i, j\) block of \([a_{i,j}]_\tau\) is the \(t_i \times t_j\) matrix with the constant entry \(a_{i,j}\).
The following Lemma shows that the maps $\nu_T$ map positive elements to positive elements.
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**Lemma**

For $\tau = (t_1, \ldots, t_d) \in \mathcal{P}(d, k)$ and $[b_{i,j}] \in M_{r,d}(B)$. There is $[c_{i,j}] \in M_{r,k}(B)$, whose entries appear in $[b_{i,j}]$, such that

$$
\nu_\tau([b_{i,j}]^* [b_{i,j}]) = [c_{i,j}]^* [c_{i,j}].
$$

**Proof.**

For $1 \leq i \leq r$ let the $r \times k$ matrix $[c_{i,j}]$ have $i$-th row

$$
[b_{i1}, \ldots, b_{i1}, b_{i2}, \ldots, b_{i2}, \ldots, b_{id}, \ldots, b_{id}]
$$

where each element $b_{ij}$ appears repeated $t_j$ consecutive times. □
Note that the maps $\nu_\tau$ are defined even if the matrix entries are from a set, so in particular for matrices with entries from a *-semigroup $S$, and although there is no natural 'positivity' for matrices with entries in $S$ one can still use partial orderings.

**Definition**

A *-semigroup $S$ is matricially ordered, write $(S, \preceq, M)$, if there is a sequence of partially ordered sets $(M_1(S), \preceq, \ldots, M_k(S), \preceq)$, $M_k(S) \subseteq M_{k+1}(S)$ for all $k \in \mathbb{N}$, with $M_1(S) = S$, satisfying (for $[n_i] \in M_1(S)$, $k(S))$

\begin{enumerate}
  \item $[n_i] \ast [n_j] = [n_i \ast n_j] \in M_k(S)$
  \item if $[a_{ij}] \preceq [b_{ij}]$ in $M_k(S)$ then $[n_i \ast a_{ij} n_j] \preceq [n_i \ast b_{ij} n_j]$ in $M_k(S)$
  \item the maps $\nu_\tau: M_d(S) \to M_k(S)$ are order maps for all $\tau \in \text{P}(d, k)$.
\end{enumerate}
Note that the maps \( \iota_\tau \) are defined even if the matrix entries are from a set, so in particular for matrices with entries from a \(*\)-semigroup \( S \), and although there is no natural 'positivity' for matrices with entries in \( S \) one can still use partial orderings.

**Definition**

A \(*\)-semigroup \( S \) is matricially ordered, write \((S, \preceq, \mathcal{M})\), if there is a sequence of partially ordered sets \((\mathcal{M}_k(S), \preceq)\), \(\mathcal{M}_k(S) \subseteq \mathcal{M}_k(S)^{sa} (k \in \mathbb{N})\), with \(\mathcal{M}_1(S) = S^{sa}\), satisfying (for \([n_i] \in \mathcal{M}_{1,k}(S)\))

a. \([n_i]^*[n_j] = [n_i^*n_j] \in \mathcal{M}_k(S)\)

b. if \([a_{i,j}] \preceq [b_{i,j}]\) in \(\mathcal{M}_k(S)\) then \([n_i^*a_{i,j}n_j] \preceq [n_i^*b_{i,j}n_j]\) in \(\mathcal{M}_k(S)\)

c. the maps \(\iota_\tau : \mathcal{M}_d(S) \rightarrow \mathcal{M}_k(S)\) are order maps for all \(\tau \in \mathcal{P}(d, k)\).
The lemma above showed that a C*-algebra $B$ has a matricial order where $\mathcal{M}_k(B)$ is the usual partially ordered set $\mathcal{M}_k(B)^{sa}$.
The lemma above showed that a C*-algebra $B$ has a matricial order where $\mathcal{M}_k(B)$ is the usual partially ordered set $\mathcal{M}_k(B)^{sa}$.

We may define *-maps $\beta : S \to T$ of matricially ordered *-semigroups $S$ and $T$ that are complete order maps - so $\beta_k : \mathcal{M}_k(S) \to \mathcal{M}_k(T)$ is defined, and an order map of partially ordered sets. A completely positive map of C*-algebras is then a complete order map.

A complete order representation of a matricially ordered *-semigroup $S$ into a C*-algebra is a *-homomorphism which is a complete order map.
\section*{C*-algebras of $S$}

If $F$ is a specified collection of *-representations of $S$ in C*-algebras, for example *-representations, contractive *-representations, or complete order *-representations, then the universal C*-algebra of $S$ is a C*-algebra $C^*_F(S)$ along with a *-semigroup homomorphism $\iota : S \to C^*_F(S)$ in $F$ satisfying the universal property

\[
\begin{array}{ccl}
S & \downarrow \iota & \rightarrow \\
\downarrow \gamma & \in F & \\
C^*(S) & \xrightarrow{\pi_\gamma} & C
\end{array}
\]

Given $\gamma : S \to C$, $\gamma \in F$, there is a unique *-homomorphism $\pi_\gamma = \pi : C^*_F(S) \to C$ with $\pi_\gamma \circ \iota = \gamma$. 
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Given $\gamma : S \to C$, $\gamma \in F$, there is a unique *-homomorphism $\pi \gamma = \pi : C_F^*(S) \to C$ with $\pi \gamma \circ \iota = \gamma$.

For an arbitrary *-semigroup one can also form the universal C*-algebra where $F$ is the collection of contractive *-representations.
Hilbert modules

Definition
Let $\beta : S \rightarrow T$ be a $*$-map of a $*$-semigroup $S$ to a matricially ordered $*$-semigroup $(T, \preceq, \mathcal{M})$. The map $\beta_k$ has the Schwarz property for $k$, if

$$
\beta_k([n_i])^* \beta_k([n_j]) \preceq \beta_k([n_i]^*[n_j])
$$

in $\mathcal{M}_k(T)$ for $[n_i] \in \mathcal{M}_{1,k}(S)$. Here $\beta_k([n_i])^* \beta_k([n_j])$ is the selfadjoint element $[\beta(n_i)^* \beta(n_j)]$ in $\mathcal{M}_k(T)$. 

A $*$-homomorphism $\sigma : S \rightarrow T$ of $*$-semigroups has the Schwarz property (since $\sigma_k([n_i])^* \sigma_k([n_j]) = \sigma_k([n_i]^*[n_j])$ for $[n_i] \in \mathcal{M}_{1,k}(S)$).

Note that if $\beta : R \rightarrow S$ and $\sigma : S \rightarrow T$ are complete order maps, $\beta$ with the Schwarz property and $\sigma$ a $*$-semigroup homomorphism, then $\sigma \beta$ is a complete order map with the Schwarz property.
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A (complete) Schwarz map to a C*-algebra $C$ is necessarily completely positive:

**Definition**

A *-map $\beta : S \rightarrow C$ from a *-semigroup $S$ into a C*-algebra $C$ is completely positive if the matrix $[\beta(n_i^*n_j)]$ is positive in $M_k(C)$ for any finite set $n_1, ..., n_k$ in $S$. 
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Completely positive maps yield Hilbert modules; so for $\beta : S \to C$ completely positive from a *-semigroup $S$ into a C*-algebra $C$ then $X = \mathbb{C}[S] \otimes_{alg} C$ has a C valued (pre) inner product (for $x = s \otimes c$, $y = t \otimes d$, with $s, t \in S$, $c, d \in C$

set $\langle x, y \rangle = \langle c, \beta(s^* t)d \rangle = c^* \beta(s^* t)d$).

After moding out by 0 vectors and completing obtain a right Hilbert module $E_C$. 
In general there is a well defined left action of $S$ on the dense submodule $X/ \sim$ of the Hilbert module $\mathcal{E}_C$, although not necessarily by adjointable, or even bounded, operators.
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Assume there is a *-map $\alpha : S \rightarrow S$ which is a complete order map satisfying the Schwarz inequality for all $k \in \mathbb{N}$.
In general there is a well defined left action of $S$ on the dense submodule $X/ \sim$ of the Hilbert module $E_C$, although not necessarily by adjointable, or even bounded, operators.

Assume there is a $\ast$-map $\alpha : S \to S$ which is a complete order map satisfying the Schwarz inequality for all $k \in \mathbb{N}$.

Then since $\iota : S \to C^*((S, \preceq, M))$ is a complete order representation, the composition $\beta = \iota \circ \alpha : D_1 \to C^*((D_1, \preceq, M))$ is a complete order map satisfying the (complete) Schwarz inequality.
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The map $\beta$ is therefore completely positive and we can form the Hilbert module $E_{C^*((S, \preceq, \mathcal{M}))}$. 
Furthermore, if the left action of $S$ extends to an action by adjointable maps on the Hilbert module $E_C$, and if

$$l : S \to \mathcal{L}(E_{C^*((S,\preceq,M))})$$

is additionally a complete order representation of the matricially ordered $*$-semigroup $S$, the universal property yields a $*$-representation

$$\phi : C^*((S,\preceq,M)) \to \mathcal{L}(E_{C^*((S,\preceq,M))})$$

defining a correspondence $E$ over the C*-algebra $C^*((S,\preceq,M))$. 
There is a *-semigroup $D_1$ for which one can describe an ordering, and matricial ordering, where the steps in this process hold. It is nonunital, and not left cancellative, so existing procedures for forming C*-algebras from semigroups, which seem largely motivated by versions of a 'left regular representation', do not apply.
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The three universal C*-algebras $C_F^*(S)$ for the three families $F$ of contractive *-representations, order representations, and complete order representations are not (canonically) isomorphic.
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The three universal C*-algebras $C_F^*(S)$ for the three families $F$ of contractive *-representations, order representations, and complete order representations are not (canonically) isomorphic.

A relative Cuntz-Pimsner C*-algebra associated with the above C*-correspondence over the C*-algebra $C^*((D_1, \preceq, \mathcal{M}))$ is isomorphic to the universal C*-algebra $\mathcal{P}$ generated by a partial isometry.
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For example with \(S\) the single element \(*\)-semigroup consisting of the identity, and \(\alpha\) the only possible map on \(S\), this process yields the universal C*-algebra generated by a unitary. The orderings play no role here.
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For example with $S$ the single element *-semigroup consisting of the identity, and $\alpha$ the only possible map on $S$, this process yields the universal C*-algebra generated by a unitary. The orderings play no role here.

Let $S$ be the two element unital (unit $u$) two element *-semigroup \{u, s\} with $s$ a selfadjoint idempotent and $\alpha$ the map sending both elements to $u$. The above Cuntz-Pimsner algebra over the C*-algebra of this semigroup is the universal C*-algebra generated by an isometry.
The free *-semigroup generated by a single element is $A_c \cong \mathbb{N}^+ \ast \mathbb{N}^-$, it consists of reduced words of nonzero integers $(n_0, n_1, ..., n_k)$ alternating in sign, multiplication is concatenation, and $(n_0, n_1, ..., n_k)^* = (-n_k, -n_{k-1}, ..., -n_0)$. 
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The *-semigroup \( A \) is a quotient of \( A_c \). Form the equivalence relation generated by the relation

\[(n_0, n_1, ..., n_k) \sim (n_0, n_1, ..., n_{i-1} \pm 1 + n_{i+1}, ... n_k)\]

whenever \( n_i = \pm 1 \) for \(1 \leq i \leq k - 1\).
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The map $\alpha : A \rightarrow A$ is defined by $\alpha(n) = (-1)n(1)$. The elements $(-1, 1)$ and $(1, -1)$ of $A^0$ are idempotents, and $\alpha(1, -1)) = (-1, 1)$.
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The *-semigroup $D_1$ is the smallest $\alpha$-closed (*-)subsemigroup of $A$ containing the element $(1, -1)$.