What is the minimal critical exponent of quasiperiodic words?

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Quasiperiodicity?


**Definition**

\( w \) is \( q \)-quasiperiodic if \( w \neq q \) (finite case) and \( w \) can be obtained by concatenations and overlaps of \( q \).
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**Examples**

- \( k \)-powers: \( qq \ldots q \) \( k \)times
  \[ abaabaabaaba \]

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  - \( k\) times
  - \( abaabaabaaba \)
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**Examples**

- \( k \)-powers: \( qq \ldots q \) \[abaabaabaaba\]
  \[aba \overbrace{abaaba}aba \overbrace{abababa}aba\]
  \[abacaba \overbrace{abacaba}cab \overbrace{abacaba}\]

Critical exponent?

Fractional power

\[ x^{\frac{p}{q}} = x^n y \text{ with } n = \lfloor \frac{p}{q} \rfloor, \quad q = |x| \text{ and } y \text{ prefix of } x \text{ of length } p - nq \]

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\begin{align*}
ababa &= (ab)^{5/2} \\
abaabaab &= (aba)^{8/3}
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Critical exponent?

**Fractional power**

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**Critical exponent of \( w \)**

\[ E(w) = \sup\{ k \in \mathbb{Q} \mid w \text{ contains a } k\text{th power} \} \]

\[ E(\text{Thue-Morse}) = 2 \]

\[ E(\text{Fibonacci}) = 2 + \phi \]
Question

\[ \min \{ E(w) \mid w \text{ quasiperiodic} \} ? \]
Reformulation of the question

Question

\[
\min \{ E(w) \mid w \text{ quasiperiodic} \}?
\]

Observation

\[ w \text{ quasiperiodic } \Rightarrow E(w) > 2. \]
Indeed \( w \) contains an overlap of \( q \) or \( q^2 \).
For all $\epsilon > 0$, over a 3-letter alphabet, there exists an infinite word with critical exponent less than $2 + \epsilon$.

So the question holds only on binary alphabets:

Is the smallest exponent $\frac{7}{3}$? $\frac{5}{2}$? $\frac{8}{3}$? other?
For all $\epsilon > 0$, over a 3-letter alphabet, there exists an infinite word with critical exponent less than $2 + \epsilon$.

So the question holds only on binary alphabets: Is the smallest exponent $\frac{7}{3}$?

Recent idea (friday)

- to use Karhumäki, Shallit 1994 and their 21-uniform morphism:

$\Rightarrow \frac{7}{3}$
# Ideas for the 7-letter alphabet

## Step 1

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Mapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$\mapsto x y z x y x$</td>
</tr>
<tr>
<td>$b$</td>
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</tr>
<tr>
<td>$c$</td>
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For all infinite word $w$, $f(w)$ is $x y z x y x$-quasiperiodic.
Ideas for the 7-letter alphabet

**Step 1**

\[
f \begin{cases} 
  a &\mapsto xyzxyzx \\
  b &\mapsto xyzzy \\
  c &\mapsto xyz 
\end{cases}
\]

for all infinite word \(w\), \(f(w)\) is \(xyzxyzx\)-quasiperiodic

**Step 2**

Choose:

- \(w, y\) and \(z\) square-free
- \(x\) letter, \(x \notin \text{alph}(yz), \text{alph}(y) \cap \text{alph}(z) = \emptyset\)

Maximal runs of exponent > 2 are:

\[
xyxyx
\]

\[f(ba) = xyzxyzxyxzzxyx\]
Ideas for the 7-letter alphabet (continue)

Consequence of Step 2

\[ E(w) = \max\left(2 + \frac{1}{1 + |y|}, 2 + \frac{1}{1 + \frac{|z|}{2 + |y|}}\right) \]

Final step

\(y\) and \(z\) can be chosen on disjoint 3-letter alphabets such that

\[ E(w) \leq 2 + \epsilon \]
Use following square-free Brandenburg’s morphism (1983) twice:

\[
\begin{align*}
    a_1 & \mapsto aba\ cabins\ cac\ bab\ cba\ cbc \\
    a_2 & \mapsto aba\ cab\ cac\ bac\ aba\ cbc \\
    a_3 & \mapsto aba\ cab\ cac\ bca\ bcb\ abc \\
    a_4 & \mapsto aba\ cab\ cba\ cab\ acb\ abc \\
    a_5 & \mapsto aba\ cab\ cba\ cbc\ acb\ abc
\end{align*}
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from 7-letter alphabet to 3-letter alphabet

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\]

with following extensions for the first time:

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\begin{align*}
a_6 \mapsto & \ dbd \ cdb \ cdc \ bdb \ cbd \ cbc, \\
a_7 \mapsto & \ ebe \ ceb \ cec \ beb \ cbe \ cbc
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If \( w \) has a run of period \( p \) and exponent \( 2 + \epsilon \) with \( \epsilon > 0 \), then \( f(w) \) has a run of exponent \( 2 + \epsilon + 17/p \)
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(In the construction on 7 letter alphabet, we can prove periods of repetitions of exponent at least 2 are \( > |xyz| \).)
Recent idea to go from 3-letter alphabet to 2-letter alphabet

Use paper by Karhumäki and Shallit in 1994 and their morphism:

\[
\begin{align*}
    a & \mapsto 011010011001001101001 \\
    b & \mapsto 100101100100110010110 \\
    c & \mapsto 100101100110110010110 \\
    d & \mapsto 011010011011001101001
\end{align*}
\]

KS1994: If \( w \) is square-free:

- \( f(w) \) contains no square \( yy \) with \( |y| > 13 \);
- \( f(w) \) contains no \( \frac{7}{3}^+ \)-powers.

It seems that taking suitable \( w \) quasiperiodic over \( \{a, b, c\} \) with exponent \( 2 < E(w) < \frac{7}{3} \), we can get \( E(f(w)) = \frac{7}{3} \).
Theorem (Karhumäki, Shallit 1994)

Let $x$ be a word avoiding $\alpha$-powers, with $2 < \alpha \leq \frac{7}{3}$. Let $\mu$ be the Thue–Morse morphism. Then there exist $u, v$ with $u, v \in \{\varepsilon, 01, 00, 11\}$ and a word $y$ avoiding $\alpha$-powers, such that $x = u\mu(y)v$. 
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Consequence:
for $w$ infinite avoiding such $\alpha$-powers, $n \geq a$, $w = u\mu^n(w')$ with $w'$.
$w$ $q$-quasiperiodic + $n$ such that $3|q| \leq |\mu^n(a)|$: contradiction.
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Consequence:
for $w$ infinite avoiding such $\alpha$-powers, $n \geq a$, $w = u\mu^n(w')$ with $w'$. $w$ q-quasiperiodic + $n$ such that $3|q| \leq |\mu^n(a)|$: contradiction.

$E(w) \geq \frac{7}{3}$
Characterization of quasiperiodic-free morphism?
That is \( w \) non-quasiperiodic \( \Rightarrow f(w) \) non-quasiperiodic.
Another problem

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| Question | If $f$ does not preserve non-quasiperiodic words, then exists $uv^\omega$ non-quasiperiodic with $f(uv^\omega)$ non-quasiperiodic? |

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Quasiperiodic infinite words
Characterization of quasiperiodic-free morphism?
That is $w$ non-quasiperiodic $\Rightarrow f(w)$ non-quasiperiodic.

They are prefix and suffix.

If $f$ does not preserve non-quasiperiodic words, then exists $uv\omega$ non-quasiperiodic with $f(uv\omega)$ non-quasiperiodic?

What about bounds on $|u|$ and $|v|$?