KK-lifting Problem and Order Structures on K-groups

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joint with George A. Elliott
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- To determine which KK-elements can be realized by a $\ast$-homomorphism makes sense at its own in KK-theory.
- Such a lifting problem is closely related to the classification of C*-algebras: when the approximate (asymptotic) unitary equivalence classes of homomorphisms are determined by their induced KK-classes, for the corresponding existence theorem, we need to lift a KK-class to a homomorphism.
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- To look for criterion for KK-lifting.
- For classification, try to connect the criterion to an invariant of C*-algebras, i.e., an order structure on the K-groups.
Cuntz’s picture of KK-groups:

**Definition**

For two C*-algebras \( A \) and \( B \), define \( KK(A, B) \) to be the homotopy classes of quasi-homomorphisms from \( A \) to \( B \), where a quasi-homomorphism is a pair of homomorphisms \( \phi_{\pm} : A \rightarrow M(B \otimes K) \) with \( \phi_+(a) - \phi_-(a) \in B \otimes K \).
The most powerful theorem for KK-groups is the Universal Coefficient theorem due to J. Rosenberg and C. Schochet:
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**Theorem**

*Let $A$ be a $C^*$-algebra in the Bootstrap class, and $B$ be a separable $C^*$-algebra, then the following sequence is exact:*

\[ 0 \rightarrow \text{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \rightarrow KK(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B)) \rightarrow 0. \]
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Examples:

- **Finite dimensional C*-algebras, Interval algebras:** from UCT, the KK-group of two such algebras $A$ and $B$ is just $\text{Hom}(K_0(A), K_0(B))$. The order structure is the usual one induced by projections.

- **Circle algebras:** from UCT, the KK-group of two such algebras $A$ and $B$ is $\text{Hom}(K_0(A), K_0(B)) \oplus \text{Hom}(K_1(A), K_1(B))$. Elliott introduced an order structure on $K_* = K_0 \oplus K_1$. There is an alternative picture due to Dadarlat and Nemethi:
  \[ K_*^+(A) := \{ ([\varphi(1)], [\varphi(e^{2\pi it})]) | \varphi \text{ is a homomorphism from } C(S^1) \text{ to } M_k(A) \text{ for some } k \}. \]
Examples (continued): what else should we look at? Torsion $K_1$ group.
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Dimension drop algebra $I_n$ and $\tilde{I}_n$,

$$I_n = \{ f \in C([0,1], M_n) \mid f(0) = 0, f(1) = \lambda 1, \lambda \in \mathbb{C} \}.$$

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For UCT, this time, we have a nontrivial Ext. part, so situation is not as same as before.
Indeed,

$$K^+(\tilde{I}_n) = \{(a, \bar{b}) \mid a \geq 1\} \cup \{(0, 0)\}.$$ 

Then any multiple (not exceeding $n$) of the KK-element $[\delta_1] - [\delta_0]$ preserves this order, but can not be lifted to a homomorphism.
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**Definition**

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for any C*-algebra \( P \) in the Bootstrap class such that \( K_0(P) = 0 \) and \( K_1(P) = \mathbb{Z}/p\mathbb{Z} \).

\[ K_0(A; \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) := KK(\widetilde{P}, A). \]

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**Definition**

The order structure is defined as follows:

\[ K_0^+(A; \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) := \{ ([\varphi(1)], [\varphi|_{I_p}]) \mid \varphi \in \text{Hom}(\tilde{I}_p, M_k(A)) \}. \]
Lemma

There is a natural short exact sequence of groups:

\[ K_0(A) \xrightarrow{\times p} K_0(A) \xrightarrow{\mu_{A;p}} K_0(A; \mathbb{Z}_p) \xrightarrow{\nu_{A;p}} K_1(A) \xrightarrow{\times p} K_1(A). \]

where \( p \geq 2 \), \( \mu_{A;p}, \nu_{A;p} \) are the Bockstein operations defined by the Kasparov product with the element of \( KK(I_p, \mathbb{C}) \) given by the evaluation \( \delta_1 : I_p \to \mathbb{C} \) and the element of \( KK^1(\mathbb{C}, I_p) \) given by the inclusion \( i : SM_p \to I_p \) respectively.
Given a KK-element $\alpha \in KK(A, B)$, it induces the following commutative diagram:

\[
\begin{array}{cccccc}
K_0(A) & \xrightarrow{\times p} & K_0(A) & \xrightarrow{\mu_{A;p}} & K_0(A; \mathbb{Z}_p) & \xrightarrow{\nu_{A;p}} & K_1(A) & \xrightarrow{\times p} & K_1(A) \\
\downarrow K_0(\alpha) & & \downarrow K_0(\alpha; \mathbb{Z}_p) & & \downarrow K_1(\alpha) & & \\
K_0(B) & \xrightarrow{\times p} & K_0(B) & \xrightarrow{\mu_{B;p}} & K_0(B; \mathbb{Z}_p) & \xrightarrow{\nu_{B;p}} & K_1(B) & \xrightarrow{\times p} & K_1(B) \\
\end{array}
\]
Then, S. Eilers realized KK-elements on the K-groups with coefficients above, and obtained the following criterion for KK-lifting of classical dimension drop interval algebras:

\[ \text{Theorem} \]

Given natural numbers \( n \), \( m \), and \( p \), with \( n \) dividing \( p \), if \( \alpha \in \text{KK}(\tilde{I}_n, \tilde{I}_m) \) induces a positive homomorphism on the K-groups with coefficients above, then \( \alpha \) can be lifted to a homomorphism.
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Generalized dimension drop interval algebras
Jiang and Su investigated the following dimension drop interval algebras:

Definition
A generalized dimension drop interval algebra $I_{[m_0, m, m_1]}$ is of the following form:

$I_{[m_0, m, m_1]} = \{ f \in C([0, 1], M_{m_0}(C)) : f(0) = a_0 \otimes \text{id}_{m_0}, f(1) = \text{id}_{m_1} \otimes a_1 \}$,

where $a_0$ and $a_1$ belong to $M_{m_0}(C)$ and $M_{m_1}(C)$ respectively, and $m_0, m_1$ divide $m$.

Question: what is going on for the KK-lifting problem of these dimension drop interval algebras? Is the order structure above enough?

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**Answer:** No.
Jiang and Su gave the following criterion for KK-lifting between two such algebras:

Theorem

The natural map $\eta: \text{KK}(A, B) \to \text{Hom}(K_0(B), K_0(A))$ is an isomorphism, where $\eta$ is the Kasparov product with K-homology elements.

A KK-element $\alpha$ can be lifted to a homomorphism if and only if $\eta(\alpha)$ is positive on the K-homology groups, where the positive cone is the subset of all K-homology classes of finite dimensional representations.

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For a natural number \(p \geq 2\) with \(m\) dividing \(p\), denote \(\mathbb{Z} \oplus \mathbb{Z}_p\) by \(G_p\), then we have \(K_0(A_m; G_p) = \mathbb{Z} \oplus Z(m, p)\), where

\[
Z(m, p) = \left\{ (\bar{b}, \bar{c}) \in \mathbb{Z}_p \oplus \mathbb{Z}_p \mid \frac{m}{m_1} c - \frac{m}{m_0} b \in p\mathbb{Z} \right\}.
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K_0^+(A_m; G_p) = \{(a, \bar{b}, \bar{c}) \in \mathbb{Z} \oplus \mathbb{Z}(m, p) \mid am_0 \geq \bar{b}, am_1 \geq \bar{c}\}.
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The Bockstein operations are given by

\[
\mu_{A_m; p} = \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \text{ and } \nu_{A_m; p} = \left( -\frac{m}{pm_0}, \frac{m}{pm_1} \right).
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**Lemma**

Given a generalized dimension drop interval algebra \( A_m \), and a positive integer \( p \) with \( m \mid p \), then each element of the Dadarlat-Loring positive cone of \( K_0(A_m; \mathbb{Z} \oplus \mathbb{Z}_p) \) can be written as a linear combination of \([\delta_0],[\delta_1],[id],[\overline{id}]\) with non-negative integer coefficients.
Given $A_m = I[m_0, m, m_1]$ and $B_n = I[m_0, n, m_1]$, realize the KK-data on K-groups with coefficients, we have:
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**Theorem**

*Given positive integers $n, m$ and $p$ with $m|p$, then the canonical map*

$$\Gamma : KK(A_m, B_n) \rightarrow \text{Hom}(K(A_m; p), K(B_n; p))$$

*is an isomorphism.*
For \( m_0, m_1 \), we always choose \( \beta_0 \geq 0 \), and \( \beta_1 \leq 0 \), such that
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\beta_0 m_0 + \beta_1 m_1 = 1.
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**Theorem (Structure of Hom($K(A_m; p), K(B_n; p)$))**

Each element $\Phi = (x, \rho, y)$ in $\text{Hom}(K(A_m; p), K(B_n; p))$ with $K_0$-multiplicity $x$ and $K_1$-multiplicity $y$ is of the following form:

$$\Phi = (x, \sigma, y) + d(0, \begin{pmatrix} -m_1 m_0 & m_0 m_0 \\ -m_1 m_1 & m_0 m_1 \end{pmatrix}, 0).$$

(⋆)

where $\sigma = \begin{pmatrix} x m_0 \beta_0 + \frac{mym_1 \beta_1}{n} & x m_0 \beta_1 - \frac{mym_0 \beta_1}{n} \\ x m_1 \beta_0 - \frac{mym_1 \beta_0}{n} & x m_1 \beta_1 + \frac{mym_0 \beta_0}{n} \end{pmatrix}$, and $d$ is an integer with $0 \leq d < \frac{n}{m m_0 m_1}$. 

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From the two theorems above, if we assume the $K_1$-multiplicity of given KK-element $\alpha$ is zero, then $\alpha$ must have the following form:

$$\alpha = (\beta_0 x - dm_1)\delta_0 + (\beta_1 x + dm_0)\delta_1.$$
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**Theorem**

*Given a KK-element $\alpha$ as above with $0 \leq d < \frac{m}{m_0 m_1 m_0}$ or $0 \leq d < \frac{m}{m_0 m_1 m_1}$, if $\Gamma(\alpha; p)$ preserves the Dadarlat-Loring order structure, then $\beta_0 x - dm_1 \geq 0$.***
Theorem

Given a $KK$-element $\alpha \in KK(A_m, B_n)$ with $K_1$-multiplicity zero, if the $K_0$-multiplicity $\times \geq m$, then $\alpha$ can be lifted to a homomorphism between the algebras.
To obtain the exact conditions under which $\alpha$ preserves the Dadarlat-Loring order, we simplify further by assuming $d = 0$, then we have:

\[ \text{Theorem} \]

Given $\alpha = \beta_0 x \delta_0 + \beta_1 x \delta_1$, let $R$ be the remainder of $\beta_0 m_0 x$ divided by $m$, and $S$ be the remainder of $\beta_0 m_0 m_1 x$ divided by $m$.

Then $\Gamma(\alpha; p)$ preserves the Dadarlat-Loring order structure if and only if:

1. $x = 0$ or $\beta_0 m_0 x \geq m$,
2. $\beta_0 m_0 m_1 x \geq m$,
3. $m_0 x \geq R$,
4. $m_1 x \geq S$.
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\[
\beta_0 m_0 m_0 x \geq m, \quad \beta_0 m_0 m_1 x \geq m
\]  

(1) \hspace{2cm} \text{(2)}

\[
m_0 x \geq R, \quad m_1 x \geq S.
\]
Note that not only in the statement above, but also in the proof, the whole requirements for preserving the Dadarlat-Loring order are included in the first part coefficient involving $\beta_0$; the negative number $\beta_1$ (with proper choices of $K_0$-multiplicity) provides flexibility for $\alpha$ not to be representable by a homomorphism.
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**Concrete Examples:** Given $m_0 = 2$, $m_1 = 3$, and $m = 12$, by the first inequality above, take $x = 2$, then the second inequality is also satisfied. We get $\alpha = 4\delta_0 - 2\delta_1$, which can not be representable by a $\ast$-homomorphism, since the minimal relation between $\delta_0$ and $\delta_1$ is $6\delta_0 = 4\delta_1$. Another example could be $x = 5$, then we get $\alpha = 10\delta_0 - 5\delta_1 = 4\delta_0 - \delta_1$. 
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Another example could be $x = 5$, then we get

$$\alpha = 10\delta_0 - 5\delta_1 = 4\delta_0 - \delta_1.$$
This result shows that the existence theorem with Dadarlat-Loring order structure fails at the level of building blocks. If one puts some restrictions on the inductive limit algebras, e.g., real rank zero, then the dynamical behaviour of the connecting maps can be controlled, we can still get a classification in terms of Dadarlat-Loring order.
Thank you!