A surface-aware projection basis for oceanic flows

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Motivation

High-resolution numerical modelling and satellite observations suggest ocean turbulence is in a surface quasi-geostrophic regime near the surface.

Surface vorticity

Baroclinic instability with $b_y \neq 0$ (left) and with $b_y = 0$ (right) (Roullet et al, JPO, 2012)
Recall quasi-geostrophic model:

\[ \partial_t q + \partial(\psi, q) = 0, \quad \text{and} \quad \partial_t b + \partial(\psi, b) = 0 \quad \text{at} \quad z = z^\pm, \]

with the inversion

\[ \partial_{xx} \psi + \partial_{yy} \psi + \partial_z \left( \frac{f^2}{N^2} \partial_z \psi \right) = q \quad \text{and} \quad \partial_z \psi = b/f \quad \text{at} \quad z = z^\pm. \]

Three dynamical variables:

- potential vorticity \( q(x, y, z, t) \),
- surface and bottom buoyancy \( b(x, y, z^\pm, t) \).

Simplified models:

- QG turbulence: \( b(x, y, z^\pm, t) = \text{const.} \),
- SQG turbulence: \( q = 0 \).
Interior and surface motion

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Predictions:

<table>
<thead>
<tr>
<th></th>
<th>QG</th>
<th>SQG</th>
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</thead>
<tbody>
<tr>
<td>energy spectrum</td>
<td>$k^{-3}$</td>
<td>$k^{-5/3}$</td>
</tr>
<tr>
<td>SSH spectrum</td>
<td>$k^{-5}$</td>
<td>$k^{-11/3}$</td>
</tr>
<tr>
<td>Rossby number</td>
<td>$k^0$</td>
<td>$k^{2/3}$</td>
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</tbody>
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Spectra in primitive equation simulations (Klein et al, JPO, 2009)
Interior and surface motion

Observed SSH: SQG $k^{-11/3}$ spectrum in energetic regions.

Le Traon et al (JPO, 2009)

Xu and Fu (JPO, 2011, 2012)
Interior and surface motion

Vertical structure of SQG motion:

$$\hat{q} = 0 \implies \partial_z \left( \frac{f^2}{N^2} \partial_z \hat{\psi} \right) - \kappa^2 \hat{\psi} = 0 \implies \hat{\psi} \propto e^{N\kappa z/f}$$

for Fourier mode \((k, l)\) with \(\kappa^2 = k^2 + l^2\).

- **Exponential decay** from surface,
- **non-zero surface buoyancy** \(b(z^\pm) = f \partial_z \hat{\psi}(z^\pm) \neq 0\).

A difficulty:

Vertical structure of SQG motion is poorly represented by standard basis of barotropic + baroclinic modes.
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Modal expansion

Standard basis of baroclinic modes:

Eigenfunctions of

\[
\left( \frac{f^2}{N^2} \psi_n' \right)' = -\lambda_n^2 \psi_n, \quad \text{with} \quad \psi_n' = 0 \text{ at } z = 0, -H.
\]

For constant \( N \):
\[
\psi_n \sim \cos(n \pi z/H), \quad n = 0, 1, \ldots
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Advantages:
- orthogonal basis, \( \int_{-H}^{0} \psi_n \psi_m \, dz \propto \int_{-H}^{0} \nabla \psi_n \cdot \nabla \psi_m \, dz \propto \delta_{mn} \),
- diagonalise energy,
- describes (interior) QG dynamics with a few modes,
- mode structure independent of \( \kappa \).

Heavily used: projection of data, basis for simplified models...
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Standard basis

**Difficulty:**

- basis unsuitable to describe SQG-like motion since \( f \psi'_n = b = 0 \) at \( z = 0, -H \),
- non-uniform convergence for surface modes
  \[ e^{N\kappa z/f} = \sum_n A_n \cos(n\pi z/H), \]
- many modes needed to represent motion with surface activity.

Need to find an alternative, ‘surface-aware’ basis.

**Some attempts:**

- Tulloch & Smith (JAS, 2009), Lapeyre (JPO, 2009): add SQG mode \( e^{-N\kappa z/f} \) to standard basis,
- Scott & Furnival (JPO, 2012): add barotropic mode to ‘Dirichlet basis’ satisfying \( \psi_n = 0 \) at \( z = 0 \).

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But, non-orthogonal, overcomplete bases.
We derive new surface-aware, orthogonal bases.

Ideas:

- Think of $Q = (q, b^+, b^-)$ not $\psi$ as the dynamical variable to be expanded,
- Recall linear algebra: a unique basis diagonalises 2 quadratic forms $x^T A x$ and $x^T B x$ (solve $Ax = \lambda Bx$),
- Choose as quadratic form conserved quantities: energy and ‘generalised enstrophy’,

$$\int_{-H}^{0} |\nabla \psi|^2 dz \quad \text{and} \quad \int_{-H}^{0} q^2 dz + \alpha_+(b^+)^2 + \alpha_-(b^-)^2.$$

Family of bases parameterised by $\alpha_\pm$. 
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**Notes:**

- **New bases**
- **Modal expansion**
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- **Introduction**
- **Conclusion**

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New bases

Basis vectors: eigenfunctions of

$$\left( \frac{f^2}{N^2} \psi_n' \right)' = -\lambda_n^2 \psi_n, \quad \text{with} \quad \frac{f^2}{N^2} \psi_n' = \pm \frac{\lambda_n^2 + \kappa^2}{\alpha_\pm} \psi_n \quad \text{at} \quad z = 0, -H.$$

Limiting cases:

- $\alpha_\pm \to \infty$: reduces to standard baroclinic basis for $n = O(1)$,
- $\alpha_\pm \to 0$: ‘Dirichlet basis’ with $\psi_n = 0$ at $z = 0, -H$
  + 2 SQG modes ($q = 0$) and imaginary $\lambda_n$. 

![Graphs showing basis functions](attachment:graph.png)
New bases

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New bases

Application

3 QG simulations of baroclinic instability:
(1) interior BC1,
(2) surface Eady,
(3) mixed Ocean.

$\alpha_\rightarrow \infty$
New bases

Choosing $\alpha_+$: maximise energy content of first 2 modes
Conclusion

- Effects of surface buoyancy gradients cannot be ignored in ocean turbulence,
- Eddies have rich, surface-intensified vertical structure that is not well-represented by standard vertical modes,
- New bases presented can capture most energy in such flows with a small truncation set,
- New bases can be very simple:

\[
\psi_0 \propto \cosh \left[ N \kappa (z + H)/f \right], \quad \psi_n \propto \sin \left[ (n - 1/2) \pi z/H \right].
\]

- New bases depend on \( \kappa \): coupling of horizontal and vertical structures.