Dynamics, dimension and classification of $\mathcal{C}^*$-algebras

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Dimension and C*-algebraic regularity

Dynamic versions of dimension and regularity
DEFINITION
Let $X$ be locally compact and metrizable. We say $X$ has dimension at most $n$, $\dim X \leq n$, if the following holds:
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For any open cover $\mathcal{V}$ of $X$, there is a finite open cover $(U_\lambda)_{\lambda \in \Lambda}$ such that $(U_\lambda)_{\lambda \in \Lambda}$ refines $\mathcal{V}$ and for each $i \in \{0, \ldots, n\}$, the $(U_\lambda)_{\lambda \in \Lambda(i)}$ are pairwise disjoint.
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For any open cover $\mathcal{V}$ of $X$, there is a finite open cover $\left( U_\lambda \right)_{\lambda \in \Lambda}$ such that

- $\left( U_\lambda \right)_{\lambda \in \Lambda}$ refines $\mathcal{V}$
- $\Lambda = \Lambda^{(0)} \cup \ldots \cup \Lambda^{(n)}$ and for each $i \in \{0, \ldots, n\}$, the $\left( U_\lambda \right)_{\Lambda(i)}$ are pairwise disjoint.
DEFINITION (W–Zacharias)
Let $A$ be a C*-algebra, $n \in \mathbb{N}$. We say $A$ has nuclear dimension at most $n$, $\dim_{\text{nuc}} A \leq n$, if the following holds:
**DEFINITION (W–Zacharias)**

Let $A$ be a $C^*$-algebra, $n \in \mathbb{N}$. We say $A$ has nuclear dimension at most $n$, $\dim_{\text{nuc}} A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is an approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$
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with $F$ finite dimensional, $\psi$ c.p.c., $\varphi$ c.p. and

$$\varphi \circ \psi = \mathcal{F, \varepsilon} \text{id}_A,$$
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and such that $F$ can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with c.p.c. order zero maps

$$\varphi^{(i)} := \varphi|_{F^{(i)}}.$$
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Let $A$ be unital. $A$ has covering number at most $n$, if the following holds:
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Let $A$ be unital. $A$ has covering number at most $n$, if the following holds:
For any $k \in \mathbb{N}$ there are c.p.c. order zero maps

$$
\phi^{(i)} : M_k \oplus M_{k+1} \to A, \ i \in \{0, \ldots, n\},
$$

such that

$$
\sum_{i=0}^{n} \phi^{(i)} (1_k \oplus 1_{k+1}) \geq 1_A.
$$
**DEFINITION/PROPOSITION** (using Toms–W, Rørdam–W)
A C*-algebra $A$ is $\mathcal{Z}$-stable if and only if for every $k \in \mathbb{N}$ there are c.p.c. order zero maps

$$
\Phi : M_k \to A_\infty \cap A'
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and

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such that

$$\Psi(e_{11}) = 1 - \Phi(1_{M_k})$$

and

$$\Phi(e_{11})\Psi(e_{22}) = \Psi(e_{22})\Phi(e_{11}) = \Psi(e_{22}).$$
DEFINITION

A unital simple $C^*$-algebra $A$ has tracial $m$-comparison, if whenever $0 \neq a, b \in M_\infty(A)_+$ satisfy

$$d_\tau(a) < d_\tau(b)$$

for all $\tau \in T(A)$, then

$$a \precsim b^{\oplus m+1}.$$
THEOREM (by many hands)

Let

\[ \mathcal{E} = \{ C(X) \rtimes_\alpha \mathbb{Z} \mid X \text{ compact, metrizable, infinite, } \alpha \text{ induced by a uniquely ergodic, minimal homeomorphism} \}. \]
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For any \( A \in E \), \( \dim_{\text{nuc}} A < \infty \iff A \text{ is } \mathbb{Z}-\text{stable} \iff A \text{ has tracial } m\text{-comparison for some } m \in \mathbb{N} . \]
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Moreover, the regularity properties ensure classification by ordered \( K \)-theory in this case.
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Moreover, the regularity properties ensure classification by ordered \( K \)-theory in this case. (Countable structures are sufficient for classification since \( T(A) \) is a singleton for each \( A \).)
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For any $L \in \mathbb{N}$, there is a system $(U(i)_l | i \in \{0, \ldots, n\}, l \in \{1, \ldots, L\})$ of open subsets such that $\alpha_1(U(i)_l) = U(i)_l + 1$ for $i \in \{0, \ldots, n\}$, $l \in \{1, \ldots, L - 1\}$ for each fixed $i \in \{0, \ldots, n\}$ the sets $U(i)_l$ are pairwise disjoint $(U(i)_l | i \in \{0, \ldots, n\}, l \in \{1, \ldots, L\})$ is an open cover of $X$. 
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- $(U_l^{(i)} \mid i \in \{0, \ldots, n\}, l \in \{1, \ldots, L\})$ is an open cover of $X$. 
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For any open cover $\mathcal{U}$ of $X$ and any $L \in \mathbb{N}$, there is a system

$$\left( U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\} \right)$$

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For any open cover $\mathcal{U}$ of $X$ and any $L \in \mathbb{N}$, there is a system

$$(U^{(i)}_{k,l} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})$$

of open subsets such that

1. $\alpha_1(U^{(i)}_{k,l}) = U^{(i)}_{k,l+1}$ for $i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L-1\}$
2. for each fixed $i \in \{0, \ldots, n\}$ the sets $U^{(i)}_{k,l}$ are pairwise disjoint.
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- For each fixed $i \in \{0, \ldots, n\}$ the sets $U_{k,l}^{(i)}$ are pairwise disjoint
- $(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})$ is an open cover of $X$ refining $\mathcal{U}$.
**DEFINITION** Let \( X \) be compact, metrizable, infinite, and \( \alpha : \mathbb{Z} \acts X \) an action. We say \((X, \mathbb{Z}, \alpha)\) has dynamic dimension at most \( n \),\n\( \dim(X, \mathbb{Z}, \alpha) \leq n \), if the following holds:

For any open cover \( \mathcal{U} \) of \( X \) and any \( L \in \mathbb{N} \), there is a system

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(U^{(i)}_{k,l} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})
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of open subsets such that

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- \((U^{(i)}_{k,l} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})\) is an open cover of \( X \) refining \( \mathcal{U} \).

**REMARK** We think of \( n + 1 \) as the number of colors, of \( K^{(i)} \) as the number of towers of color \( i \), and of \( L \) as the length of the towers.
DEFINITION

Let \((X, Z, \alpha)\) be a compact dynamical system, \(m \in \mathbb{N}\) and \(U, V \subset X\) open subsets.

We say \(U \preceq_m V\), if the following holds:

For any compact subset \(Y \subset U\), there are a system of open subsets of \(Y\)
\[
(U_i^k)_{i \in \{0, \ldots, m\}, k \in \{1, \ldots, K(i)\}}
\]
and a system of open subsets of \(V\)
\[
(V_i^k)_{i \in \{0, \ldots, m\}, k \in \{1, \ldots, K(i)\}}
\]
such that

\[
\text{for each } i, k \text{ there is } r(i)^k \text{ with } \alpha^{r(i)^k}(U_i^k) \subset V_i^k
\]

\[
\text{for each fixed } i, \text{ the sets } V_i^k \text{ are pairwise disjoint}
\]

\[
\text{the } U_i^k \text{ cover all of } Y.
\]
DEFINITION

Let \((X, Z, \alpha)\) be a compact dynamical system, \(m \in \mathbb{N}\) and \(U, V \subset X\) open subsets.
We say \(U\) is \(m\)-dominated by \(V\), \(U \preceq_m V\), if the following holds:

For any compact subset \(Y \subset U\), there are a system of open subsets of \(Y\) \((U(i)_k | i \in \{0, \ldots, m\}, k \in \{1, \ldots, K(i)_k\})\) and a system of open subsets of \(V\) \((V(i)_k | i \in \{0, \ldots, m\}, k \in \{1, \ldots, K(i)_k\})\) such that:

1. For each \(i, k\), there is \(r(i)_k\) with \(\alpha r(i)_k(U(i)_k) \subset V(i)_k\)
2. For each fixed \(i\), the sets \(V(i)_k\) are pairwise disjoint
3. The \(U(i)_k\) cover all of \(Y\).
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Let \((X, Z, \alpha)\) be a compact dynamical system, \(m \in \mathbb{N}\) and \(U, V \subset X\) open subsets. We say \(U\) is \(m\)-dominated by \(V\), \(U \preceq_m V\), if the following holds: For any compact subset \(Y \subset U\), there are a system of open subsets of \(Y\)

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(U_k^{(i)} \mid i \in \{0, \ldots, m\}, k \in \{1, \ldots, K^{(i)}\})
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For any compact subset \(Y \subset U\), there are a system of open subsets of \(Y\)
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and a system of open subsets of \(V\)
\[ (V^{(i)}_k \mid i \in \{0, \ldots, m\}, k \in \{1, \ldots, K^{(i)}\} ) \]
such that
- for each \(i, k\) there is \(r^{(i)}_k\) with \(\alpha^{(i)}_r(U^{(i)}_k) \subset V^{(i)}_k\)
**DEFINITION**

Let $(X, Z, \alpha)$ be a compact dynamical system, $m \in \mathbb{N}$ and $U, V \subset X$ open subsets.

We say $U$ is $m$-dominated by $V$, $U \preceq_m V$, if the following holds:

For any compact subset $Y \subset U$, there are a system of open subsets of $Y$

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such that

- for each $i, k$ there is $r_k^{(i)}$ with $\alpha_{r_k^{(i)}}(U_k^{(i)}) \subset V_k^{(i)}$
- for each fixed $i$, the sets $V_k^{(i)}$ are pairwise disjoint
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For any compact subset $Y \subset U$, there are a system of open subsets of $Y$
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such that
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▶ for each fixed $i$, the sets $V_k^{(i)}$ are pairwise disjoint
▶ the $U_k^{(i)}$ cover all of $Y$. 
DEFINITION

We say \((X, \mathbb{Z}, \alpha)\) (\(\alpha\) minimal) has dynamic \(m\)-comparison, if, whenever \(U, V \subset X\) are open subsets with \(\mu(U) < \mu(V)\) for any regular invariant Borel probability measure \(\mu\) on \(X\), then \(U \preceq_m V\).
DEFINITION

Let \((X, \mathbb{Z}, \alpha)\) be a compact dynamical system. We say \((X, \mathbb{Z}, \alpha)\) is dynamically \(\mathbb{Z}\)-stable, if the following holds:

For any \(K \in \mathbb{N}\), there are systems \((V_j, k | j, k \in \{1, \ldots, K\})\) and \((U_k | k \in \{1, \ldots, K\})\) of open subsets of \(X\) such that:

1. The sets \(\bigcup k V_j, k\) are pairwise disjoint for \(1 \leq j \leq K\).
2. \(\alpha_1(V_j, k) = \alpha_1(V_j, k + 1)\) for each \(1 \leq j \leq K\) and \(1 \leq k \leq K - 1\).
3. \(\alpha_1(U_k) = \alpha_1(U_{k+1})\) for each \(1 \leq k \leq K - 1\),
4. \(V_j, k \sim V_{j+1}, k\) for each \(1 \leq j \leq K - 1\) and \(1 \leq k \leq K\).

Thus, \(X = \bigcup j V_j, k \cup U_k \triangleright U_1 \preceq V_1, 1\).
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- \(\alpha_1(U_k) = \alpha_1(U_{k+1})\) for each \(1 \leq k \leq K - 1\)
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- \(V_{j,k} \sim V_{j+1,k}\) for each \(1 \leq j \leq K - 1\) and \(1 \leq k \leq K\)
- for each fixed \(k\), \(X = \bigcup_j V_{j,k} \cup U_k\)
DEFINITION

Let \((X, \mathbb{Z}, \alpha)\) be a compact dynamical system.
We say \((X, \mathbb{Z}, \alpha)\) is dynamically \(\mathbb{Z}\)-stable, if the following holds:

For any \(K \in \mathbb{N}\), there are systems

\[(V_{j,k} \mid j, k \in \{1, \ldots, K\})\] and \((U_k \mid k \in \{1, \ldots, K\})\)

of open subsets of \(X\) such that

- the sets \(\bigcup_k V_{j,k}\) are pairwise disjoint for \(1 \leq j \leq K\)
- \(\alpha_1(V_{j,k}) = \alpha_1(V_{j,k+1})\) for each \(1 \leq j \leq K\) and \(1 \leq k \leq K - 1\)
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- for each fixed \(k\), \(X = \bigcup_j V_{j,k} \cup U_k\)
- \(U_1 \preceq V_{1,1}\).
THEOREM

Let \( X \) be compact, metrizable, infinite, and \( \alpha : \mathbb{Z} \curvearrowright X \) minimal.
THEOREM

Let $X$ be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ minimal. If $(X, \mathbb{Z}, \alpha)$ is dynamically $\mathbb{Z}$-stable, then $C(X) \rtimes_{\alpha} \mathbb{Z}$ is $\mathbb{Z}$-stable.
THEOREM

Let \((X, \mathbb{Z}, \alpha)\) be compact, metrizable, and minimal.

For the proof, one has to construct invariant measures from a system of open coverings of the form \(U(i)_k, l\) \(i \in \{0, \ldots, n\}, k \in \{1, \ldots, K(i)\}, l \in \{1, \ldots, L\}\) (as in the definition of dynamic dimension), which become finer and finer, and for which \(L\) becomes larger and larger.

For \(V \subset X\) open, \(\mu(V)\) is then defined as a limit along some ultrafilter of expressions like \(\sharp\{l | U(i)_k, l \subset V\}^L\).
THEOREM
Let \((X, \mathbb{Z}, \alpha)\) be compact, metrizable, and minimal. If \(\dim(X, \mathbb{Z}, \alpha) \leq m\), then \((X, \mathbb{Z}, \alpha)\) has \(m\)-comparison.
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THEOREM (Hirshberg–W–Zacharias, 2011)
Let \((X, \mathbb{Z}, \alpha)\) be compact, metrizable, and minimal. Suppose \(X\) is finite dimensional.

\[ \dim \text{Rok}(X, \mathbb{Z}, \alpha) \leq 2(\dim X + 1) - 1 \]
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What about more general groups?

For $\mathbb{Z}^d$, replace $\{1, \ldots, L \}$ by $\{1, \ldots, L \}^d$ in the definition of $\dim \text{Rok}(X, \mathbb{Z}^d, \alpha)$. In this case, we don't have a general theorem, but:

**EXAMPLE (Matui)**

$C^*\text{(Penrose tiling)} \cong MC(X) \rtimes \alpha \mathbb{Z}^2$, where $X$ is the Cantor set and $\alpha$ is free and minimal. $(X, \mathbb{Z}^2, \alpha)$ has a factor of form $(X \times X, \mathbb{Z}^2, \alpha_1 \times \alpha_2)$ with $\alpha_1, \alpha_2$ both minimal.

From the preceding theorem we get $\dim \text{Rok}(X, \mathbb{Z}^2, \alpha) < \infty$, hence $\dim \text{Rok}(X, \mathbb{Z}^2, \alpha) < \infty$ and $\dim \text{nuc}(C^*\text{(Penrose tiling)}) < \infty$.

We do not know, however, whether this ensures classifiability.
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$C^\ast(\text{Penrose tiling}) \cong M_{C^\ast(X)} \rtimes \alpha \mathbb{Z}^2$, where $X$ is the Cantor set and $\alpha$ is free and minimal. $(X, \mathbb{Z}^2, \alpha)$ has a factor of form $(X \times X, \mathbb{Z}^2, \alpha_1 \times \alpha_2)$ with $\alpha_1, \alpha_2$ both minimal.

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$$C^*(\text{Penrose tiling}) \sim_{\text{M}} C(X) \rtimes_{\alpha} \mathbb{Z}^2,$$

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Let $G$ be a hyperbolic group acting on its Rips complex $\bar{X}$ ($G$ acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

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**THEOREM**

Let $\alpha$ be a minimal action of $G = \mathbb{Z}$ on the compact, metrizable, finite dimensional, infinite space $X$. Then, there is $d \in \mathbb{N}$ such that the following holds:

For any $L \in \mathbb{N}$ there is an open cover $U$ of $G \times X$ satisfying $\bigcup U$ has covering number (or dimension) at most $d$ for every $x \in X$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$ for every $g \in G$ and $U \in \mathcal{U}$, $gU \in \mathcal{U}$ for every $0 \neq g \in G$ and $U \in \mathcal{U}$, $gU \cap U = \emptyset$, i.e., for every $U \in \mathcal{U}$, the subgroup $G_U = \{g \in G | gU = U\}$ is trivial.
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