Generic properties of measure preserving actions

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Workshop on applications to Operator Algebras
Toronto, September 12, 2012

Polish spaces and groups

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- The group $\mathrm{Aut}(X,\mu)$ of measure-preserving bijections of a standard atomless probability space (X,μ) is a Polish group with the topology induced by the maps $T \mapsto \mu(T(A)\Delta A)$ (where A ranges over all measurable subsets of X).

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- Another example that will come up is the group $L^0(\mathbb{T})$, which is the unitary group of the abelian von Neumann algebra $L^\infty(X, \mu)$.

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Question

What does a typical element of $\mathrm{Hom}(\Gamma, G)$ look like? Which properties are *generic* in $\mathrm{Hom}(\Gamma, G)$?

The conjugacy action

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- There exists a comeager conjugacy class in $\mathrm{Hom}(\Gamma, G)$ whenever Γ is finite, and conjugacy classes are meager whenever Γ is amenable and infinite (Glasner–Weiss 2005).

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- It is an open problem whether conjugacy classes are meager for all infinite Γ.

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Assume that $\Delta \leq \Gamma$ are countable groups. How do the generic properties in $\mathrm{Hom}(\Delta, G)$ relate to the generic properties in $\mathrm{Hom}(\Gamma, G)$?

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Note that the restriction map is obviously category-preserving when $\Delta = \mathbb{F}_n \leq \mathbb{F}_m = \Gamma$ (it is open).

An extension of the Kuratowki–Ulam theorem

Theorem (M.–Tsankov)

Let X, Y be Polish spaces, and $f : X \to Y$ be a continuous, category-preserving map. Then the following are equivalent, for $A \subseteq X$ Baire–measurable:

- A is comeager in X.

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In symbols:

$$\left(\forall^* x \in X \ A(x)\right) \Leftrightarrow \left(\forall^* y \in Y \ \forall^* z \in f^{-1}(\{y\}) \ A(z)\right) .$$

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The classical Kuratowski–Ulam theorem corresponds to the case where f is a projection map.

Restrictions of measure-preserving actions

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Let Γ be a countable abelian group and Δ be an infinite cyclic subgroup. Then a generic measure-preserving Δ-action extends to a *free* Γ-action.

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Let Γ be a countable abelian group and Δ be an infinite cyclic subgroup. Then the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu)) \to \mathrm{Hom}(\Delta, \mathrm{Aut}(\mu))$ is category-preserving.

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Thus, under the above assumptions on $\Delta \leq \Gamma$, whenever a generic Δ action satisfies some property (P), the restriction to Δ of a generic Γ-action also satisfies property (P).

Restrictions of measure-preserving actions II

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The map $\phi_n \colon \begin{cases} G \to G \\ g \mapsto g^n \end{cases}$ is category-preserving for all $n \geq 1$ (In particular, a generic element of G admits roots of all orders).

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How far can these results be pushed?

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Using the structure theorem for finitely-generated abelian groups, and the extension of the Kuratoski–Ulam theorem mentioned above, one can prove the following.

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Let Γ be a countable abelian group and Δ be a *finitely generated* subgroup. Then the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ is category-preserving.

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O. Ageev has recently announced a negative answer.

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There exist a polycyclic group Γ and an infinite cyclic subgroup $\Delta \leq \Gamma$ such that a generic measure-preserving Δ-action does not extend to a measure-preserving Γ-action.

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The proof of the above observation depends on another result of King (1986): the closed subgroup generated by a generic element of G is maximal abelian; equivalently, the centralizer of a generic element g of G is equal to the closure of $\{g^n \colon n \in \mathbb{Z}\}$.

A new proof of King's result on centralizers of generic elements I.

Now we describe a simple proof of King's result on centralizers of generic elements (note: King's original result is actually stronger, as it applies to all elements of rank 1). The proof is extracted from the proof of a more general result in a joint work with T. Tsankov.

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For H a Polish group, we identify $\mathrm{Hom}(\mathbb{Z}^2, H)$ with

$$\mathcal{C}(H) = \{(a,b) \in H \colon ab = ba\} \ .$$

For $h \in H$ $\mathcal{C}(h)$ denotes the centralizer of h.

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Then pick $c \in O_1 \cap A$: we have $(c, c^n) \in O$ and $\pi(c, c^n) = c \in A$. □

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Since $\overline{\langle a \rangle}$ is obviously closed in $\mathcal{C}(a)$, we get $\mathcal{C}(a) = \overline{\langle a \rangle}$ for a generic $a \in H$. □

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Let M be a separable, diffuse von Neumann algebra. Then any maximal abelian subalgebra of M is diffuse, so its unitary group is isomorphic to $L^0(X,\mu)$.

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Of course, a maximal abelian subgroup of $\mathcal{U}(M)$ must be the unitary group of a masa.

Separable von Neumann algebras II.

To sum up:

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The same result holds for $\mathcal{U}(\ell_2)$; this was originally proved by Todor Tsankov and myself, but one can give a simpler proof based on the technique discussed above and the notion of extreme amenability.

Extreme amenability is a G_δ property.

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Let Γ be a countable group, and H be a Polish group. Then

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is G_δ in $\mathrm{Hom}(\Gamma, H)$.

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Corollary (M.–Tsankov)

A generic element of $\mathcal{U}(\ell_2)$ generates a closed subgroup isomorphic to $L^0(X, \mu)$.

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We saw that a generic element of $\mathrm{Aut}(X, \mu)$ generates a closed subgroup which is maximal abelian and extremely amenable; similar ideas can also be used to proved that this subgroup is always generically monothetic.

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Question

Is it true that a generic element of $\mathrm{Aut}(X, \mu)$ generates a closed subgroup isomorphic to $L^0(X, \mu)$?

What about $\mathrm{Aut}(X, \mu)$?

We saw that a generic element of $\mathrm{Aut}(X, \mu)$ generates a closed subgroup which is maximal abelian and extremely amenable; similar ideas can also be used to proved that this subgroup is always generically monothetic.

More is kown:

Theorem (Solecki)

For a generic $g \in \mathrm{Aut}(X, \mu)$, the closed subgroup generated by g is a continuous homomorphic image of $L^0(X, \mu)$, and contains an increasing chain of finite-dimensional tori whose union is dense.

Question

Is it true that a generic element of $\mathrm{Aut}(X, \mu)$ generates a closed subgroup isomorphic to $L^0(X, \mu)$?

Thank you for your attention!