Generic properties of measure preserving actions

Julien Melleray

Institut Camille Jordan (Université de Lyon)

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**Definition**

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Examples

- The group $\text{Aut}(X, \mu)$ of measure-preserving bijections of a standard atomless probability space $(X, \mu)$ is a Polish group with the topology induced by the maps $T \mapsto \mu(T(A)\Delta A)$ (where $A$ ranges over all measurable subsets of $X$).
Polish spaces and groups

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• Another example that will come up is the group \( L^0(\mathbb{T}) \), which is the unitary group of the abelian von Neumann algebra \( L^\infty(X, \mu) \).
**Notation**

$\Gamma$ will always denote a countable discrete group, and $G$ will stand for $\text{Aut}(X, \mu)$. 

**Definition**

The space of homomorphisms $\text{Hom}(\Gamma, G)$ is a closed subset of $G \Gamma$, hence a Polish space.

We may think of $\text{Hom}(\Gamma, G)$ as the space of actions of $\Gamma$ on $(X, \mu)$.

**Question**

What does a typical element of $\text{Hom}(\Gamma, G)$ look like? Which properties are generic in $\text{Hom}(\Gamma, G)$?
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- Hence any Baire-measurable, conjugacy-invariant subset of $\text{Hom}(\Gamma, G)$ must be either meager or comeager.
- There exists a comeager conjugacy class in $\text{Hom}(\Gamma, G)$ whenever $\Gamma$ is finite, and conjugacy classes are meager whenever $\Gamma$ is amenable and infinite (Glasner–Weiss 2005).
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- There exists a comeager conjugacy class in $\text{Hom}(\Gamma, G)$ whenever $\Gamma$ is finite, and conjugacy classes are meager whenever $\Gamma$ is amenable and infinite (Glasner–Weiss 2005).
- It is an open problem whether conjugacy classes are meager for all infinite $\Gamma$. 
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Let $f : X \to Y$ be a continuous map. Say that $f$ is *category-preserving* if $f^{-1}(O)$ is comeager in $X$ whenever $O$ is comeager in $Y$ (e.g. any open map is category-preserving).
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Question (revisited)
Assume that $\Delta \leq \Gamma$ are countable groups. When is the restriction map $\text{Res} : \text{Hom}(\Gamma, G) \to \text{Hom}(\Delta, G)$ category-preserving?

Note that the restriction map is obviously category-preserving when $\Delta = F_n \leq F_m = \Gamma$ (it is open).
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Theorem (M.–Tsankov)

Let $X, Y$ be Polish spaces, and $f : X \rightarrow Y$ be a continuous, category-preserving map. Then the following are equivalent, for $A \subseteq X$ Baire–measurable:

- $A$ is comeager in $X$.
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In symbols:

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The classical Kuratowski–Ulam theorem corresponds to the case where \( f \) is a projection map.
Restrictions of measure-preserving actions

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Corollary (equivalent reformulation of Ageev’s theorem)
Let $\Gamma$ be a countable abelian group and $\Delta$ be an infinite cyclic subgroup. Then the restriction map $\text{Res}: \text{Hom}(\Gamma, \text{Aut}(\mu)) \to \text{Hom}(\Delta, \text{Aut}(\mu))$ is category-preserving.
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Thus, under the above assumptions on $\Delta \leq \Gamma$, whenever a generic $\Delta$ action satisfies some property (P), the restriction to $\Delta$ of a generic $\Gamma$-action also satisfies property (P).
Ageev’s result is based on an earlier result of King, corresponding to the case when $\Gamma = \mathbb{Z}$ and $\Delta = n\mathbb{Z}$. 
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**Theorem (King 2000)**

The map $\phi_n : \begin{cases} G \to G \\ g \mapsto g^n \end{cases}$ is category-preserving for all $n \geq 1$ (In particular, a generic element of $G$ admits roots of all orders).
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At roughly the same time as Ageev, Tikhonov also obtained similar results (for instance the fact that the restriction map from $\text{Hom}(\mathbb{Z}^d, G)$ to $\text{Hom}(\mathbb{Z}, G)$ preserves category).
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How far can these results be pushed?
Using the structure theorem for finitely-generated abelian groups, and the extension of the Kuratowski–Ulam theorem mentioned above, one can prove the following.

**Theorem (M.)**

Let $\Gamma$ be a countable abelian group and $\Delta$ be a finitely generated subgroup. Then the restriction map $\text{Res}: \text{Hom}(\Gamma, G) \to \text{Hom}(\Delta, G)$ is category-preserving.

**Question**

Can one remove the assumption that $\Delta$ is finitely generated in the previous theorem?

O. Ageev has recently announced a negative answer.
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**Observation (M.)**

There exist a polycyclic group $\Gamma$ and an infinite cyclic subgroup $\Delta \leq \Gamma$ such that a generic measure-preserving $\Delta$-action does not extend to a measure-preserving $\Gamma$-action.
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The proof of the above observation depends on another result of King (1986): the closed subgroup generated by a generic element of $G$ is maximal abelian; equivalently, the centralizer of a generic element $g$ of $G$ is equal to the closure of $\{g^n : n \in \mathbb{Z}\}$.
Now we describe a simple proof of King’s result on centralizers of generic elements (note: King’s original result is actually stronger, as it applies to all elements of rank 1). The proof is extracted from the proof of a more general result in a joint work with T. Tsankov.
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**Notation**

For $H$ a Polish group, we identify $\text{Hom}(\mathbb{Z}^2, H)$ with

$$C(H) = \{(a, b) \in H : ab = ba\}.$$

For $h \in H$ $C(h)$ denotes the centralizer of $h$. 

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**Lemma**

Let $H$ be a Polish group such that $\{(a, b) \in C(H) : b \in \langle a \rangle\}$ is dense in $C(H)$. Then the map $\pi : \{C(H) \to H : (a, b) \mapsto a\}$ is category-preserving.
A new proof of King’s result on centralizers of generic elements I.

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Proof.
Let $A$ be a dense subset of $H$; enough to prove that $\pi^{-1}(A)$ is dense in $C(H)$. So let $O$ be nonempty open in $C(H)$ and assume w.l.o.g. that

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Then pick $c \in O_1 \cap A$: we have $(c, c^n) \in O$ and $\pi(c, c^n) = c \in A$. \qed
Theorem
Assume again that $H$ is a Polish group such that
\{$(a,b) \in C(H): b \in \langle a \rangle$\} is dense in $C(H)$. Then the centralizer of a
generic element $h$ of $H$ is equal to $\langle h \rangle$. 

Note that the assumption of this theorem is easily seen to be satisfied
when $H = \text{Aut}(X,\mu)$. 

Proof.
We have $\forall (a,b) \in C(H)$ $b\in \langle a \rangle$.
Applying the fact that $(a,b) \mapsto a$ is category-preserving from
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Since $\langle a \rangle$ is obviously closed in $C(a)$, we get $C(a) = \langle a \rangle$ for a generic
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The strategy of proof above is fairly flexible. As pointed out by my student F. Le Maître, it is easy to see the following.

Lemma

Let $M$ be a separable von Neumann algebra. Then 

$$\{ (a, b) \in \mathbb{C} \left(\mathcal{U}(M) \right) : b \in \langle a \rangle \}$$

is dense in $\mathbb{C} \left(\mathcal{U}(M) \right)$.

Thus, a generic element in the unitary group of a separable von Neumann algebra always generates a maximal abelian subgroup.

Lemma

Let $M$ be a separable, diffuse von Neumann algebra. Then any maximal abelian subalgebra of $M$ is diffuse, so its unitary group is isomorphic to $L^0(X, \mu)$.

Of course, a maximal abelian subgroup of $\mathcal{U}(M)$ must be the unitary group of a masa.
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**Lemma**

Let $M$ be a separable von Neumann algebra. Then $\{(a, b) \in C(U(M)) : b \in \langle a \rangle\}$ is dense in $C(U(M))$. 

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To sum up:

**Theorem (Le Maître)**

Let $M$ be a diffuse separable von Neumann algebra; a generic element of $\mathcal{U}(M)$ generates a closed subgroup which is maximal abelian and isomorphic to $L^0(X, \mu)$. 

The same result holds for $\mathcal{U}(\ell^2)$; this was originally proved by Todor Tsankov and myself, but one can give a simpler proof based on the technique discussed above and the notion of extreme amenability.
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Extreme amenability is a $G_\delta$ property.

**Definition**
Recall that a topological group $H$ is *extremely amenable* if any continuous action of $H$ on a compact space has a fixed point.

Theorem (M.–Tsankov)
Let $\Gamma$ be a countable group, and $H$ be a Polish group. Then 

\[ \{ \pi \in \text{Hom}(\Gamma, H) : \pi(\Gamma) \text{ is extremely amenable} \} \]

is $G_\delta$ in $\text{Hom}(\Gamma, H)$.

Theorem (M.–Tsankov)
In both $\text{Aut}(X, \mu)$ and $U(\ell^2)$, a generic element generates an extremely amenable subgroup.

Corollary (M.–Tsankov)
A generic element of $U(\ell^2)$ generates a closed subgroup isomorphic to $L^0(X, \mu)$.
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Extreme amenability is a $G_δ$ property.

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J. Melleray  
Generic properties of measure-preserving actions
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Thank you for your attention!