Universality in classes of Banach spaces and compact spaces

Piotr Koszmider

Polish Academy of Sciences
Outline

A. Abstract nonsense
1. Types of universality
2. Mappings
3. Dualities
4. Associations between classes of compact spaces and classes of Banach spaces
5. Examples of classes of compact and Banach spaces

B. The existence and non-existence of universal spaces
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Types of universality

Consider a class of structures $C$ and two classes of mappings, some injections called embeddings and some surjections called quotient maps.

A structure $X \in C$ is called injectively universal iff for every $Y \in C$ there is an embedding $f: Y \to X$.

A structure $X \in C$ is called surjectively universal iff for every $Y \in C$ there is a quotient map $f: X \to Y$.

A structure $X \in C$ is called weakly universal iff for every $Y \in C$ there is $Z \in C$ and an embedding $f: X \to Z$ and a quotient $g: Y \to Z$. 

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- Consider a class of structures $\mathcal{C}$ and two classes of mappings, some injections called embeddings and some surjections called quotient maps.
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Mappings

For classes of Boolean algebras

- Embeddings: injective Boolean homomorphisms
- Quotients: surjective Boolean homomorphisms

For classes of compact spaces

- Embeddings: injective continuous mappings
- Quotients: surjective continuous mappings

For classes of Banach spaces

- Isomorphic package:
  - Embeddings: injective linear continuous mappings
  - Quotients: surjective linear continuous mappings
- Isometric package:
  - Embeddings: linear isometries (norm preserving mappings)
  - Quotients: compositions of the quotient maps with isometries

Terminology

- For Boolean algebras and Banach spaces: universal = injectively universal
- For compact spaces: universal = surjectively universal
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Dualities

The Stone duality

\[ \text{K} \to \{ f \in C(K) \mid f : K \to \{0, 1\}\} = \text{Clop}(K) \]

\[ \text{A} \to \{ h \in \text{hom}(A) \mid h : A \to \{0, 1\}\} = \text{Stone}(A) \]

\[ [a] = \{ h \in K^A : h(a) = 1\} \]

\[ \text{K} \to \text{Clop}(\text{Stone}(A)) \equiv A, \text{Stone}(\text{Clop}(K)) \equiv K \]

A similar construction

\[ \text{K} \to \{ f \in C(K) \mid f : K \to \mathbb{R}\} = C(K) \]

\[ \text{X} \to \{ h \in X^* \mid h : X \to \mathbb{R}, ||h|| \leq 1\} = B_{X^*} \]

Semidualities:

\[ C(B_{X^*}) \supseteq X, B(C(K)^*) \supseteq K \]
Dualities

- The Stone duality
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The Stone duality

- $K \to \{ f \in C(K) | f : K \to \{0, 1\}\} = Clop(K)$
The Stone duality

- $K \rightarrow \{ f \in C(K) \mid f : K \rightarrow \{0, 1\} \} = \text{Clop}(K)$
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$$[a] = \{ h \in K_{\mathcal{A}} : h(a) = 1 \}$$

- $\operatorname{Clop}(\operatorname{Stone}(\mathcal{A})) \equiv \mathcal{A}, \operatorname{Stone}(\operatorname{Clop}(K)) \equiv K$
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  - $K \to \{ f \in C(K) \mid f : K \to \{0, 1\} \} = Clop(K)$
  - $\mathcal{A} \to \{ h \in hom(\mathcal{A}) \mid h : \mathcal{A} \to \{0, 1\} \} = Stone(\mathcal{A})$
  
  $$[a] = \{ h \in K_\mathcal{A} : h(a) = 1 \}$$

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- A similar construction
  - $K \to \{ f \in C(K) \mid f : K \to \mathbb{R} \} = C(K)$
  - $X \to \{ h \in X^* \mid h : X \to \mathbb{R}, ||h|| \leq 1 \} = B_{X^*}$

  $$[f, r, \varepsilon] = \{ h \in B_{X^*} : h(f) \in (r - \varepsilon, r + \varepsilon) \}$$
Dualities

- **The Stone duality**
  - \( K \rightarrow \{ f \in C(K) \mid f : K \rightarrow \{0, 1\} \} = \text{Clop}(K) \)
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- **Semidualities:** \( C(B_{X^*}) \supseteq X, B_{C(K)^*} \supseteq K \)
Dualities

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\[ K \rightarrow \{ f \in C(K) \mid f : K \rightarrow \mathbb{R} \} = C(K) \]
\[ X \rightarrow \{ h \in X^* \mid h : X \rightarrow \mathbb{R}, ||h|| \leq 1 \} = B_{X^*} \]

\[ [f, r, \varepsilon] = \{ h \in B_{X^*} : h(f) \in (r - \varepsilon, r + \varepsilon) \} \]

Semidualities: \( C(B_{X^*}) \supseteq X, B_{C(K)^*} \supseteq K \)
Suppose that $K$ and $B$ as above are associated.

- If $K$ is universal for $K$, then $C(K)$ is isometrically universal for $B$.
- If there is a universal Banach space $X$ for $B$, then $B(X^*)$ is universal for $B$ as well.
- If $X$ is weakly universal for $B$, then $B(X^*)$ is weakly universal for $K$. 

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Suppose $\mathcal{K}$ is a class of compact spaces and $\mathcal{B}$ is a class of Banach spaces. We say that they are associated iff

1. $\mathcal{K} \in \mathcal{K} \implies \mathcal{C}(\mathcal{K}) \in \mathcal{B}$
2. $X \in \mathcal{B} \implies \mathcal{B}X^* \in \mathcal{K}$

If we have equivalences we say that the classes are strongly associated. If we have equivalence only in the first line, we say that the classes are $\mathcal{K}$-associated.

Suppose that $\mathcal{K}$ and $\mathcal{B}$ as above are associated.

1. If $\mathcal{K}$ is universal for $\mathcal{K}$, then $\mathcal{C}(\mathcal{K})$ is isometrically universal for $\mathcal{B}$
2. If there is a universal Banach space $X$ for $\mathcal{B}$, then $\mathcal{C}(\mathcal{B}X^*)$ is universal for $\mathcal{B}$ as well.
3. If $X$ is weakly universal for $\mathcal{B}$, then $\mathcal{B}X^*$ is weakly universal for $\mathcal{K}$.
Suppose $\mathcal{K}$ is a class of compact spaces and $\mathcal{B}$ is a class of Banach spaces. We say that they are associated iff
\begin{itemize}
    \item $K \in \mathcal{K} \implies C(K) \in \mathcal{B}$
\end{itemize}
If $\mathcal{K}$ is universal for $\mathcal{K}$, then $C(K)$ is isometrically universal for $\mathcal{B}$.
If there is a universal Banach space $X$ for $\mathcal{B}$, then $C(BX^*)$ is universal for $\mathcal{B}$ as well.
If $X$ is weakly universal for $\mathcal{B}$, then $BX^*$ is weakly universal for $\mathcal{K}$.
Suppose $\mathcal{K}$ is a class of compact spaces and $\mathcal{B}$ is a class of Banach spaces. We say that they are associated iff

- $K \in \mathcal{K} \Rightarrow C(K) \in \mathcal{B}$
- $X \in \mathcal{B} \Rightarrow B_{X^*} \in \mathcal{K}$

If $\mathcal{K}$ is universal for $\mathcal{K}$, then $C(K)$ is isometrically universal for $\mathcal{B}$. If there is a universal Banach space $X$ for $\mathcal{B}$, then $C(B_{X^*})$ is universal for $\mathcal{B}$ as well. If $X$ is weakly universal for $\mathcal{B}$, then $B_{X^*}$ is weakly universal for $\mathcal{K}$.
Associations

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Examples of associated classes

- $\text{UE}_\kappa$ and $\overline{\mathcal{H}}_\kappa$ are $\kappa$-associated and are not strongly associated.

- $\text{WCG}_\kappa$ and $\mathcal{A}_\kappa$ are $\kappa$-associated and are not strongly associated.

- $\text{RN}_\kappa$ and $\mathcal{A}_\kappa$ are $\kappa$-associated, strongly associated if $\kappa < \beta$ and not strongly associated if $\kappa \geq \beta$.

Assuming MA$^+$ and not CH the classes $\mathcal{C}_\kappa$ and $\text{WLD}_\kappa$ are strongly associated for any uncountable cardinal $\kappa$.

- $\text{UE}_\kappa$ and $\mathcal{B}_\kappa$ are strongly associated.
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Compact Hausdorff spaces of weight $\kappa (\mathbb{K}_\kappa)$ and Banach spaces of density $\leq \kappa (\mathbb{B}_\kappa)$
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- (Parovichenko; 1963) Assume CH. Then $\mathbb{N}^*$ is universal in $\mathbb{K}_{2^\omega}$ and so $\ell_\infty/c_0 \equiv C(\mathbb{N}^*)$ is isometrically universal in $\mathbb{B}_{2^\omega}$.

- Let $\kappa$ be a cardinal. $[0,1]_\kappa$ is injectively universal in $\mathbb{K}_{2^\omega}$. In particular, any space that maps onto $[0,1]_{2^\omega}$, and so, $\mathbb{N}^*$, is weakly universal in $\mathbb{K}_{2^\omega}$. In particular $\ell_\infty/c_0$ is isometrically weakly universal for $\mathbb{B}_{2^\omega}$.

- (Dow, Hart; 2001) It is consistent that there is no universal compact space for $\mathbb{K}_{2^\omega}$.

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Questions

Are any of the statements below equivalent in ZFC:

- There is a universal Banach space of density \( \leq 2^\omega \).
- There is an isometrically universal Banach space of density \( \leq 2^\omega \).
- There is a universal compact space of weight \( \leq 2^\omega \).

Is every universal Banach space of density \( \leq 2^\omega \) isomorphic to an isometrically universal Banach space of density \( \leq 2^\omega \)? Is it isometrically universal itself?

Is there a property \( P \) of norms such that:

- If \((X, ||||)\) has \( P \) and \( Y \) is a closed subspace of \( X \), then \((Y, ||||\rbrack Y)\) has \( P \) as well.
- Not all norms on Banach spaces of density \( \leq 2^\omega \) have \( P \).
- \( C(K) \) spaces have equivalent norm with property \( P \).

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Universality and non-universality of $N^* = \beta \mathbb{N} \setminus \mathbb{N}$

The following spaces are always continuous images of $N^*$:

- separable spaces
- (Parovichenko; 1963) spaces of weight $\omega_1$
- (Bell, Shapiro, Simon; 1996) polyadic spaces of weight $\leq 2\omega$

The following spaces always isometrically embed into $\ell_\infty / c_0$:

- Banach spaces with weakly $\ast$ separable dual ball, in particular, all HI spaces
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- (Bell, Shapiro, Simon; 1996) polyadic spaces of weight $\leq 2^\omega$

The following spaces always isometrically embed into $\ell_\infty/c_0$:
- Banach spaces with weakly* separable dual ball, in particular, all HI spaces
- Banach spaces of density $\omega_1$
Universality and non-universality of $\ell_\infty/c_0 \equiv C(N^*)$

The following spaces may not be continuous images of $(Kunen; 1968) [0, \omega_2]...$

The following spaces may not isomorphically embed into $\ell_\infty/c_0 \equiv (Brech, Koszmider; 2012) C([0, 2\omega]) \equiv (Todorcevic; 2011)$ some first countable and some Corson compacta of weight $\leq 2\omega$

$(Brech, P. K.; 2012)$ It is consistent that there exist universal Banach spaces for $B_{2\omega}$ but $l_\infty/c_0$ is not among them.

$(Krupski, Marciszewski, 201?)$ It is consistent that a Banach space isomorphically embeds in $l_\infty/c_0$ but it does not embed isometrically.

Piotr Koszmider (Polish Academy of Sciences) Universality and Forcing Toronto, 12 11 / 26
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Questions

Is it consistent that \([0, 2\omega]\) is not a continuous image of \(N^\ast\) but \(C([0, 2\omega])\) isomorphically embeds in \(\ell_\infty/c_0\)?

Suppose that \(K\) is the Stone space of the measure algebra (Borel sets of \([0, 1]\) divided by Lebesgue measure sets zero). Is it consistent that \(C(K)\) does not isometrically embed into \(\ell_\infty/c_0\)?

Is it consistent that \(N^\ast\) is not universal for \(K_{2\omega}\) but \(\ell_\infty/c_0\) is universal for \(B_{2\omega}\)?
Questions

Is it consistent that $[0, 2^\omega]$ is not a continuous image of $\mathbb{N}^*$ but $C([0, 2^\omega])$ isomorphically embeds in $\ell_\infty/c_0$?
Questions

- Is it consistent that $[0, 2^\omega]$ is not a continuous image of $\mathbb{N}^*$ but $\mathcal{C}([0, 2^\omega])$ isomorphically embeds in $\ell_\infty/c_0$?

- Suppose that $K$ is the Stone space of the measure algebra (Borel sets of $[0, 1]$ divided by Lebesgue measure sets zero). Is it consistent that $\mathcal{C}(K)$ does not isometrically embed into $\ell_\infty/c_0$?
Questions

- Is it consistent that \([0, 2^\omega]\) is not a continuous image of \(\mathbb{N}^*\) but \(C([0, 2^\omega])\) isomorphically embeds in \(\ell_\infty/c_0\)?
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Uniform Eberlein Compact and Hilbert generated Banach spaces I

K is a uniform Eberlein compactum iff K is homeomorphic to a compact subspace of a Hilbert space with the weak topology.

X is a Hilbert generated Banach space iff there is a Hilbert space H and a bounded linear operator T: H → X with a dense range.

(Benyamini, Starbird; 1976; Fabian, Godefroy, Zizler; 2001)

UEκ and Hκ are K-associated and are not strongly associated.

Bl2(κ) is injectively universal for UEκ. In particular C(A(κ)N) is weakly universal in UEκ.

(Benyamini, Rudin, Wage; 1977)

A(κ)N is weakly universal in UEκ. In particular C(A(κ)N) is weakly universal for Hκ.

(M. Bell; 2002) There is a compact K ∈ UEω1 which is not a continuous image of any space A(κ)N. In particular the space A(κ)N is not universal in UEκ for an uncountable κ.
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(M. Bell; 2000) It is consistent that there is no universal uniform Eberlein compact space of weight $\omega_1$.

(C. Brech, P. K.; To appear in PAMS) It is consistent that there is no Banach space of density $2^{\omega}$ or $\omega_1$ which contains isomorphic copies of all Banach spaces from $\text{UE}_2^{\omega}$ or from $\text{UE}_{\omega_1}$ respectively.
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- (M. Bell; 2000) It is consistent that there is no universal uniform Eberlein compact space of weight $\omega_1$
- (M. Bell; 2000) Assume $\kappa = 2^{<\kappa}$. Then, there is a universal uniform Eberlein compact space $UE_\kappa$ in $UE_\kappa$ and so $C(UE_\kappa)$ is isometrically universal Banach space for $\mathcal{H}_\kappa$. In particular, the above results hold for $\kappa = 2^\omega = \omega_1$ if we assume the continuum hypothesis.
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Questions
Questions

- Is it consistent that there is a universal graph of size $\omega_1 (2^\omega)$ but there is no universal Banach space for $B_{\omega_1} (B_{2^\omega})$? Or any other class we consider here?
Scattered compact space and Asplund Banach spaces

A Banach space is called Asplund if dual spaces to separable subspaces are (norm) separable. A compact $K$ is scattered iff $C(K)$ is Asplund (Mazurkiewicz, Sierpiński; 1920). There is no universal scattered space of a given weight.

$d: K \times K \to \mathbb{R}^+ \cup \{0\}$ fragments $K$ iff for every $F \subseteq K$, for every $\varepsilon > 0$ there is an open $U \subseteq K$ such that $U \cap F \neq \emptyset$ and $\text{diam} \ d(U \cap F) < \varepsilon$.

(Namioka, Phelps, Jayne, Rogers) the dual norm fragments compact subsets of $B_X^*$ iff $X$ is Asplund (Szlenk 1968, Wojtaszczyk 1970, Hajek, Lancien, Montesinos 2007). For any cardinal $\kappa$ there is no universal reflexive or Asplund Banach space of density $\leq \kappa$. Piotr Koszmider (Polish Academy of Sciences) Universality and Forcing Toronto, 12 16 / 26
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A compact space is called Radon-Nikodym compact iff there is a l.s.c. metric on it which fragments it.

\[ d: K \times K \to \mathbb{R}^+ \cup \{0\} \]

is l.s.c. iff \( \{ (x, y) \in K^2 : d(x, y) \geq r \} \) is closed for every \( r \in \mathbb{R}^+ \cup \{0\} \).

A Banach space \( X \) is said to be Asplund generated iff there is an Asplund space \( A \) and a bounded linear operator \( T: A \to X \) with dense range. (I. Namioka; 1987)

\( \text{RN}_\kappa \) and \( A_\kappa \) are \( K \)-associated, (A. Avilés; 2005) strongly associated if \( \kappa < b \) and not strongly associated if \( \kappa \geq b \);
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- A Banach space \( X \) is said to be Asplund generated iff there is an Asplund space \( A \) and a bounded linear operator \( T : A \to X \) with dense range.

- (I. Namioka; 1987) \( \mathbb{RN}_\kappa \) and \( \overline{A}_\kappa \) are \( K \)-associated,
Radon-Nikodym compact space and Asplund generated Banach spaces

- A compact space is called Radon-Nikodým compact iff there is a l.s.c. metric on it which fragments it.
- $d : K \times K \to \mathbb{R}_+ \cup \{0\}$ is l.s.c. iff $\{(x, y) \in K^2 : d(x, y) \geq r\}$ is closed for every $r \in \mathbb{R}_+ \cup \{0\}$
- A Banach space $X$ is said to be Asplund generated iff there is an Asplund space $A$ and a bounded linear operator $T : A \to X$ with dense range.

(I. Namioka; 1987) $\mathbb{R}N_\kappa$ and $\overline{A}_\kappa$ are $K$-associated,

(A. Avilés; 2005) strongly associated if $\kappa < b$ and not strongly associated if $\kappa \geq b$;
Questions

Is it consistent that there are universal spaces in one of the classes $\mathbb{R}^2_\omega$, $\mathbb{R}_\omega^1$?

Is it consistent that there are universal spaces in one of the classes $\mathcal{A}_2^\omega$, $\mathcal{A}_\omega^1$?

Is it possible to associate to each Radon-Nikodym compact $K$ an ordinal index $i(K)$ having the following properties:

1. $|i(K)|$ is not bigger than the weight of $K$,
2. If $L$ is a closed subset of $K$ or if $L$ is a continuous image of $K$, then $i(L) \leq i(K)$,
3. For every $\alpha < \kappa^+$ there is a Radon-Nikodym compactum $K$ of weight $\kappa$ such that $i(K) > \alpha$.

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Questions

- Is it consistent that there are universal spaces in one of the classes \( \mathbb{RN}_{2\omega}, \mathbb{RN}_{\omega_1} \)?
Questions

- Is it consistent that there are universal spaces in one of the classes $\mathbb{RN}_{2\omega}, \mathbb{RN}_{\omega_1}$?

- Is it consistent that there are universal spaces in one of the classes $\mathcal{A}_{2\omega}, \mathcal{A}_{\omega_1}$?
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Questions

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Questions

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  - For every $\alpha < \kappa^+$ there is a Radon-Nikodým compactum $K$ of weight $\kappa$ such that $i(K) > \alpha$. 

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Questions

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- Is it possible to associate to each Radon-Nikodým compact $K$ an ordinal index $i(K)$ having the following properties:
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  - If $L$ is a closed subset of $K$ or if $L$ is a continuous image of $K$, then $i(L) \leq i(K)$,
  - For every $\alpha < \kappa^+$ there is a Radon-Nikodým compactum $K$ of weight $\kappa$ such that $i(K) > \alpha$. 
A compact space is called an Eberlein compactum iff it is homeomorphic to a compact subset of a Banach space with the weak topology.

A Banach space $X$ is called WCG iff there is $K \subseteq X$ such that

1. $K$ is compact w.r.t. the weak topology of $X$
2. $\text{lin}(K)$ is norm dense in $X$

(Amir, Lindenstrauss; 1968; Rosenthal; 1974)

$E_\kappa$ and $WCG_\kappa$, are $K$-associated and are not strongly associated.
A compact space is called an Eberlein compactum iff it is homeomorphic to a compact subset of a Banach space with the weak topology.

\[ \text{A Banach space } \mathcal{X} \text{ is called WCG iff there is } \mathcal{K} \subseteq \mathcal{X} \text{ such that:} \]

1. \( \mathcal{K} \) is compact w. r. t. the weak topology of \( \mathcal{X} \).
2. \( \operatorname{lin}(\mathcal{K}) \) is norm dense in \( \mathcal{X} \).

(Amir, Lindenstrauss; 1968; Rosenthal; 1974)

\( E \kappa \) and \( WCG \kappa \) are \( \mathcal{K} \)-associated and are not strongly associated.
A compact space is called an Eberlein compactum iff it is homeomorphic to a compact subset of a Banach space with the weak topology.

**A Banach space** $X$ **is called WCG iff there is** $K \subseteq X$ **such that**

1. $K$ is compact w. r. t. the weak topology of $X$,
2. $\text{lin}(K)$ is norm dense in $X$ (Amir, Lindenstrauss; 1968; Rosenthal; 1974)

$E^K$ and $WCG^K$ are $K$-associated and are not strongly associated.
Eberlein Compact and weakly compactly generated (WCG) Banach spaces I

- A compact space is called an Eberlein compactum iff it is homeomorphic to a compact subset of a Banach space with the weak topology.
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A Banach space $X$ is called WCG iff there is $K \subseteq X$ such that:

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2. $\text{lin}(K)$ is norm dense in $X$.

(Amir, Lindenstrauss; 1968; Rosenthal; 1974) $E_\kappa$ and $WCG_\kappa$, are $K$-associated and are not strongly associated.
Eberlein Compact and weakly compactly generated (WCG) Banach spaces II

If $\kappa \omega = \kappa$ or $\kappa = \omega_1$ then there is no weakly universal Eberlein compact of weight $\kappa$ nor a universal WCG Banach space of density $\kappa$. If $\kappa$ is a strong limit cardinal of countable cofinality, then there is a universal Eberlein compact of weight $\kappa$, and so, there is a universal WCG Banach space of density $\kappa$. 

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(Argyros, Benyamini; 1987) If $\kappa^\omega = \kappa$ or $\kappa = \omega_1$ then there is no weakly universal Eberlein compact of weight $\kappa$ nor a universal WCG Banach space of density $\kappa$. If $\kappa$ is a strong limit cardinal of countable cofinality, then there is a universal Eberlein compact of weight $\kappa$, and so, there is a universal WCG Banach space of density $\kappa$. 
Questions

Is it consistent that there is no universal space in $E_\omega$ in $WCG_\omega$?

Is it consistent that there is a universal space in $E_{\omega^2}$ in $WCG_{\omega^2}$?
Questions

- Is it consistent that there is no universal space in $E_{\omega_\omega}$, in $\mathcal{WCG}_{\omega_\omega}$?
Questions

- Is it consistent that there is no universal space in $\mathbb{E}_{\omega}$, in $\mathcal{WCG}_{\omega}$?
- Is it consistent that there is a universal space in $\mathbb{E}_{\omega_2}$, in $\mathcal{WCG}_{\omega_2}$?
Questions

- Is it consistent that there is no universal space in $E_{\omega_\omega}$, in $WCG_{\omega_\omega}$?
- Is it consistent that there is a universal space in $E_{\omega_2}$, in $WCG_{\omega_2}$?
Questions

- Is it consistent that there is no universal space in $E_{\omega_\omega}$, in $WCG_{\omega_\omega}$?
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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces I
A compact space is called a Corson compactum iff it is homeomorphic to compact subspace of the $\Sigma$-product of the unit intervals.
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A Banach space is called weakly Lindelöf iff it is Lindelöf in the weak topology
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Every WCG space is weakly Lindelöf
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces I

- A compact space is called a Corson compactum iff it is homeomorphic to compact subspace of the \( \Sigma \)-product of the unit intervals
- A Banach space is called weakly Lindelöf iff it is Lindelöf in the weak topology
- Every WCG space is weakly Lindelöf
- Every Eberlein compact space is Corson compact
A compact space is called a Corson compactum iff it is homeomorphic to compact subspace of the $\Sigma$-product of the unit intervals.

A Banach space is called weakly Lindelöf iff it is Lindelöf in the weak topology.

Every WCG space is weakly Lindelöf.

Every Eberlein compact space is Corson compact.

Corson compact space has property $M$ iff every Radon measure on it has separable support.
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces I

- A compact space is called a Corson compactum iff it is homeomorphic to compact subspace of the $\Sigma$-product of the unit intervals.
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- Every WCG space is weakly Lindelöf.
- Every Eberlein compact space is Corson compact.
- Corson compact space has property M iff every Radon measure on it has separable support.
- A Banach space $X$ is called WLD iff $X$ with the weak topology is a continuous image of a closed subset of $L^\omega_{\kappa}$. 
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

- (Argyros, Mercourakis; 1993) $K \in \mathcal{CM}_\kappa$ if and only if $\mathcal{C}(K) \in \mathcal{WLD}_\kappa$

Assuming $\text{MA} + \neg \text{CH}$ (Argyros, Mercourakis, Negrepontis; 1988) the classes $\mathcal{C}_\kappa = \mathcal{CM}_\kappa$ and $\mathcal{WLD}_\kappa$ are strongly associated for any uncountable cardinal $\kappa$.

Assuming $\text{CH}$ (Argyros, Mercourakis, Negrepontis; 1988) $\mathcal{C}_\omega$ is not associated with any class of Banach spaces.

$\mathcal{WLD}_\omega$ is not associated with any class of compact spaces.

$\mathcal{CM}_\kappa$ is associated with a class of Banach spaces for any uncountable cardinal $\kappa$;
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

- (Argyros, Mercourakis; 1993) $K \in CM_\kappa$ if and only if $C(K) \in WLD_\kappa$

- (Argyros, Mercourakis; 1993) $X \in WLD_\kappa$ if and only if $B_{X^*} \in C_\kappa$, 

Assuming $MA + \neg CH$ the classes $C_\kappa = CM_\kappa$ and $WLD_\kappa$ are strongly associated for any uncountable cardinal $\kappa$.

Assuming $CH$ (Argyros, Mercourakis, Negrepontis; 1988) $C_2^{\omega}$ is not associated with any class of Banach spaces $2^{\omega}$ (Kalenda, Plebanek; 2002) $WLD_2^{\omega}$ is not associated with any class of compact spaces $2^{\omega}$ (Kalenda, Plebanek; 2002) $L_2^{\omega}$ is not associated with any class of compact spaces. $CM_\kappa$ is associated with a class of Banach spaces for any uncountable cardinal $\kappa$.

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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

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Assuming $\text{CH}$ $\mathcal{C}_\omega = \mathcal{CM}_\omega$ is not associated with any class of Banach spaces.

$L_\omega$ is not associated with any class of compact spaces

$\mathcal{CM}_\kappa$ is associated with a class of Banach spaces for any uncountable cardinal $\kappa$. 

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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

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- Assuming $CH$

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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

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- (Plebanek, 2002) $\mathcal{CM}_\kappa$ is associated with a class of Banach spaces for any uncountable cardinal $\kappa$;
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

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Assuming $\text{CH}$

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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces II

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- (Plebanek, 2002) $\mathcal{CM}_\kappa$ is associated with a class of Banach spaces for any uncountable cardinal $\kappa$;
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces III

For every countably tight compact space $K$ of weight $\leq 2^\omega$ there is a Corson compact $K'$ of weight $\leq 2^\omega$ where all Radon measures have separable supports and which is not a continuous image of any closed set of $K$. In particular there is no universal weakly Lindelöf determined Banach spaces in $WLD_{2^\omega}$. There is no universal weakly Lindelöf Banach space of density $2^\omega$. 

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(Todorcevic; 1995) For every countably tight compact space $K$ of weight $\leq 2^\omega$ there is a Corson compact $K'$ of weight $\leq 2^\omega$ where all Radon measures have separable supports and which is not a continuous image of any closed set of $K$. In particular there is no universal weakly Lindelöf determined Banach spaces in $\mathcal{WLD}_{2^\omega}$. 
Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces III

(Todorčević; 1995) For every countably tight compact space $K$ of weight $\leq 2^\omega$ there is a Corson compact $K'$ of weight $\leq 2^\omega$ where all Radon measures have separable supports and which is not a continuous image of any closed set of $K$. In particular there is no universal weakly Lindelöf determined Banach spaces in $WLD_{2^\omega}$.

There is no universal weakly Lindelöf Banach space of density $2^\omega$. 
Questions

Is it consistent that the class $\mathcal{L}_\kappa$ of Lindelöf Banach spaces in the weak topology of density $\leq \kappa$ is associated with a class of compact spaces for an uncountable $\kappa$?

Are there (consistently) universal Banach spaces in $\text{WLD}_\kappa$ for $\kappa \neq 2^{\omega}$, for example consistently for $\kappa = \omega_1$?

Are there (consistently) universal Corson compact spaces (where all Radon measures have separable supports) spaces of weight $\leq \kappa$ for $\kappa \neq 2^{\omega}$, for example consistently for $\kappa = \omega_1$?

Are there (consistently) universal Banach spaces in $\mathcal{L}_\kappa$ for $\kappa \neq 2^{\omega}$, for example consistently for $\kappa = \omega_1$?
Questions

Is it consistent that the class $\mathcal{L}_\kappa$ of Lindelöf Banach spaces in the weak topology of density $\leq \kappa$ is associated with a class of compact spaces for an uncountable $\kappa$?
Questions

- Is it consistent that the class $\mathcal{L}_\kappa$ of Lindelöf Banach spaces in the weak topology of density $\leq \kappa$ is associated with a class of compact spaces for an uncountable $\kappa$?

- **Are there (consistently) universal Banach spaces in $\mathcal{WLD}_\kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?**
Questions

- Is it consistent that the class $\mathcal{L}_\kappa$ of Lindelöf Banach spaces in the weak topology of density $\leq \kappa$ is associated with a class of compact spaces for an uncountable $\kappa$?

- Are there (consistently) universal Banach spaces in $\mathcal{WLD}_\kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?

- Are there (consistently) universal Corson compact spaces (where all Radon measures have separable supports) spaces of weight $\leq \kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?
Questions

- Is it consistent that the class $\mathcal{L}_\kappa$ of Lindelöf Banach spaces in the weak topology of density $\leq \kappa$ is associated with a class of compact spaces for an uncountable $\kappa$?

- Are there (consistently) universal Banach spaces in $\mathcal{WLD}_\kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?

- Are there (consistently) universal Corson compact spaces (where all Radon measures have separable supports) spaces of weight $\leq \kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?

- Are there (consistently) universal Banach spaces in $L_\kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?
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http://arxiv.org/abs/1209.4294
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