A microscopic derivation
of Ginzburg–Landau theory

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The Ginzburg–Landau Model

Introduced in 1950 as a phenomenological model of a superconductor. For given vector and scalar potentials $A$ and $W$ on a compact set $C$,

$$E_{GL}(\psi) = \int_C \left[ B_1 \left| (-i \nabla + 2A(x)) \psi(x) \right|^2 + B_2 W(x) |\psi(x)|^2 + B_3 \left( |\psi(x)|^4 - 2D |\psi(x)|^2 \right) \right] dx$$

Here, $B_1, B_3 > 0$, $B_2 \in \mathbb{R}$ and $D > 0$ are coefficients.

Ginzburg–Landau energy $E_{GL} = \inf_{\psi} E_{GL}(\psi)$

A minimizing $\psi$ describes the macroscopic variations in the superfluid density. The normal state corresponds to $\psi \equiv 0$, while $|\psi| > 0$ describes superconducting particles.

For us, $C = [0, 1]^3$ and $\psi$ satisfies periodic boundary conditions.

One is often interested in minimizing over both $\psi$ and $A$, adding an additional field energy term. For us, $A$ is fixed (but arbitrary).
**The BCS Functional**

Bardeen–Cooper–Schrieffer (1957): a *microscopic theory* of superconductivity

State of the system described by a $2 \times 2$ operator-valued matrix

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \bar{\alpha} & 1 - \bar{\gamma} \end{pmatrix} \quad \text{with} \quad 0 \leq \Gamma \leq 1$$

Here, $0 \leq \gamma \leq 1$ is the 1-particle density matrix, and $\alpha$ the *Cooper-pair wavefunction*.

For chemical potential $\mu \in \mathbb{R}$ and temperature $T > 0$, the *BCS energy functional* is

$$\text{Tr} \left[ \left( \left( -i \nabla + \tilde{A}(x) \right)^2 - \mu + \tilde{W}(x) \right) \gamma \right] - T S(\Gamma) + \int \int \tilde{V}(x - y) |\alpha(x, y)|^2 dxdy.$$ 

The entropy equals $S(\Gamma) = -\text{Tr}[\Gamma \ln \Gamma]$.

The BCS functional can be heuristically derived from the full many-body Hamiltonian with pair-interaction $V$ via two steps of simplification. First, one considers only *quasi-free states*, and second one neglects the direct and exchange term in the interaction energy.
**Microscopic vs. Macroscopic Scale**

We are interested in interactions $\tilde{V}(x) = V(h^{-1}x)$ of size one varying on the **microscopic scale** and external fields $\tilde{A}(x) = hA(hx)$ and $\tilde{W}(x) = h^2W(hx)$ which are weak and vary on the **macroscopic scale**. Here $h$ is a small parameter. Thus,

$$\mathcal{F}^{BCS}(\Gamma) := \text{Tr} \left[ \left( -ih\nabla + hA(x) \right)^2 - \mu + h^2W(x) \right] \gamma - TS(\Gamma)$$

$$+ \int \int_{C \times \mathbb{R}^3} V(h^{-1}(x - y))|\alpha(x, y)|^2 \, dx \, dy$$

To avoid boundary conditions, we assume that the system is periodic (with period 1). $C$ denotes the unit cube $[0, 1]^3$, and Tr stands for the **trace per unit volume**.

We make the following **assumptions** on the functions $A$, $W$ and $V$ appearing in $\mathcal{F}^{BCS}$.

- $W$ and $A$ are periodic, and $\hat{W}(p)$ and $|\hat{A}(p)|(1 + |p|)$ are summable.

- $V$ is real-valued and reflection-symmetric, i.e., $V(x) = V(-x)$, with $V \in L^{3/2}(\mathbb{R}^3)$.

Non-local potentials $V$ (as in the original BCS model) could also be considered.
**The Translation-Invariant Case**

For $W = 0 = A$, we can restrict to translation-invariant states $\Gamma$. In this case, there exists a critical temperature $T_c \geq 0$ such that

- For $T \geq T_c$, $\alpha = 0$ in any minimizer of $\mathcal{F}^{\text{BCS}}$.
- For $T < T_c$, $\alpha \neq 0$ in any minimizer of $\mathcal{F}^{\text{BCS}}$.

In fact, $T_c$ turns out to be the unique $T$ such that

$$\frac{-\nabla^2 - \mu}{\tanh\left(\frac{-\nabla^2 - \mu}{2T}\right)} + V(x) =: K_T(-i\nabla) + V(x)$$

has 0 as its lowest eigenvalue (Hainzl, Hamza, Seiringer, Solovej 2008).

In the following, we shall **assume** that $V$ is such that $T_c > 0$, and that the eigenvalue 0 of $K_{T_c}(-i\nabla) + V$ is **simple**. This is satisfied, e.g., if $\hat{V} \leq 0$ (and not identically zero).

Let $\alpha_0$ denote the eigenfunction of $K_{T_c}(-i\nabla) + V$ corresponding to eigenvalue 0.
**Main Results: Energy Asymptotics**

Let $\Gamma_0$ denote the minimizer of $\mathcal{F}^{\text{BCS}}$ for $V = 0$, i.e.,

$$\Gamma_0 := \begin{pmatrix} \gamma_0 & 0 \\ 0 & 1 - \gamma_0 \end{pmatrix} \quad \text{with} \quad \gamma_0 = \left(1 + \exp \left((-i h \nabla + h A(x))^2 + h^2 W(x) - \mu / T\right)\right)^{-1}

Define the energy difference

$$F^{\text{BCS}}(T, \mu) = \inf_{0 \leq \Gamma \leq 1} \mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0).$$

Note that $\mathcal{F}^{\text{BCS}}(\Gamma_0) = T \text{Tr} \ln (1 - \gamma_0) = O(h^{-3})$ for small $h$.

**THEOREM 1.** Fix $D > 0$. For appropriate coefficients $B_1$, $B_2$ and $B_3$

$$F^{\text{BCS}}(T_c(1 - h^2 D), \mu) = h \left(E^{\text{GL}} + o(1)\right)$$

with $E^{\text{GL}} = \inf_{\psi} \mathcal{E}^{\text{GL}}(\psi)$ and const. $h^2 \geq o(1) \geq -\text{const. } h^{1/5}$ for small $h$.

For smooth enough $A$ and $W$, one could also expand $\mathcal{F}^{\text{BCS}}(\Gamma_0)$ to order $h$. We bound directly the energy difference, however!
THEOREM 2. If $\Gamma$ is an approximate minimizer of $\mathcal{F}^{\text{BCS}}$ at $T = T_c(1 - h^2 D)$, in the sense that

$$\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma_0) + h \left( D E^{\text{GL}} + \epsilon \right)$$

for some small $\epsilon > 0$, then the corresponding $\alpha$ can be decomposed as

$$\alpha = \frac{h}{2} (\psi(x)\tilde{\alpha}_0(-ih\nabla) + \tilde{\alpha}_0(-ih\nabla)\psi(x)) + \sigma$$

with $\mathcal{E}^{\text{GL}}(\psi) \leq E^{\text{GL}} + \epsilon + \text{const.} \ h^{1/5}$ and

$$\int_{\mathcal{C} \times \mathbb{R}^3} |\sigma(x, y)|^2 \, dx \, dy \leq \text{const.} \ h^{1-2/5}$$

To appreciate the bound on $\sigma$, note that the square of the $L^2(\mathcal{C} \times \mathbb{R}^3)$ norm of the main term in $\alpha$ is of the order $h^{-1} = h^{-3} h^2$. To leading order in $h$, we thus have

$$\alpha(x, y) \approx \frac{1}{2h^2} (\psi(x) + \psi(y)) \alpha_0 \left( \frac{x-y}{h} \right) \approx h^{-2} \psi \left( \frac{x+y}{2} \right) \alpha_0 \left( \frac{x-y}{h} \right)$$
**The Coefficients in the GL Functional**

Let $t$ be the Fourier transform of $2K_{Tc} \alpha_0 = -2V \alpha_0$, where $\|\alpha_0\|_2 = 1$. Let

$$g_1(z) = \frac{e^{2z} - 2ze^z - 1}{z^2(1 + e^z)^2}, \quad g_2(z) = g'_1(z) + 2g_1(z)/z$$

and

$$C = \left( \beta_c \int_{\mathbb{R}^3} t(q)^4 \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} \, dq \right)^{-1} \int_{\mathbb{R}^3} \frac{t(q)^2}{\cosh^2\left( \frac{\beta_c}{2}(q^2 - \mu) \right)} \, dq.$$

Then the matrix $B_1$ and the numbers $B_2$ and $B_3$ are given by

$$(B_1)_{ij} = C \frac{\beta_c^2}{16} \int_{\mathbb{R}^3} t(q)^2 \left( \delta_{ij} g_1(\beta_c(q^2 - \mu)) + 2\beta_c q_i q_j g_2(\beta_c(q^2 - \mu)) \right) \frac{dq}{(2\pi)^3},$$

$$B_2 = C \frac{\beta_c^2}{4} \int_{\mathbb{R}^3} t(q)^2 g_1(\beta_c(q^2 - \mu)) \frac{dq}{(2\pi)^3},$$

$$B_3 = C^2 \frac{\beta_c^2}{16} \int_{\mathbb{R}^3} t(q)^4 \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} \frac{dq}{(2\pi)^3}.$$
**Key Semiclassical Estimates**

For $\psi \in H^2_{\text{loc}}(\mathbb{R}^d)$ and $t$ “sufficiently nice”, let $\Delta$ denote the operator

$$\Delta = -\frac{h}{2} (\psi(x)t(-ih\nabla) + t(-ih\nabla)\psi(x))$$

The **effective Hamiltonian** on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^2$ is

$$H_\Delta = \begin{pmatrix} (-ih\nabla + hA(x))^2 - \mu + h^2W(x) & \Delta \\ \bar{\Delta} & -(ih\nabla + hA(x))^2 + \mu - h^2W(x) \end{pmatrix}$$

**THEOREM 3.** Let $f(z) = -\ln (1 + e^{-z})$ and $\varphi(p) = \frac{1}{2} \frac{t(p)}{p^2 - \mu} \tanh(\frac{\beta}{2}(p^2 - \mu))$. Then

$$\frac{h^d}{\beta} \text{Tr} [f(\beta H_\Delta) - f(\beta H_0)] = h^2 E_1 + h^4 E_2 + O(h^6) \left( \|\psi\|_{H^1_1(c)}^6 + \|\psi\|_{H^2_2(c)}^2 \right),$$

for explicit $E_1$ and $E_2$. Moreover, the off-diagonal entry $\alpha_\Delta$ of $[1 + e^{\beta H_\Delta}]^{-1}$ satisfies

$$\left\| \alpha_\Delta - \frac{h}{2} (\psi(x)\varphi(-ih\nabla) + \varphi(-ih\nabla)\psi(x)) \right\|_{H^1} \leq \text{const.} h^{3-d/2} \left( \|\psi\|_{H^2_2(c)} + \|\psi\|_{H^1_1(c)}^3 \right)$$
**Upper Bound to the Energy**

One simple takes as **trial state**

$$\Gamma = \Gamma_\Delta := [1 + e^{\beta H_\Delta}]^{-1}$$

with $t$ the Fourier transform of $2K_{T_c} \alpha_0 = -2V \alpha_0$, and computes

$$\mathcal{F}^{BCS}(\Gamma_\Delta) - \mathcal{F}^{BCS}(\Gamma_0) = -\frac{1}{2\beta} \text{Tr} \left[ \ln(1 + e^{-\beta H_\Delta}) - \ln(1 + e^{-\beta H_0}) \right]$$

$$- \hbar^{2-2d} \int_{C \times \mathbb{R}^d} V \left( \frac{x-y}{\hbar} \right) \left| \frac{1}{2} (\psi(x) + \psi(y)) \alpha_0 \left( \frac{x-y}{\hbar} \right) \right|^2 \frac{dx \, dy}{(2\pi)^d}$$

$$+ \int_{C \times \mathbb{R}^d} V \left( \frac{x-y}{\hbar} \right) \left| \frac{\hbar^{1-d}}{2(2\pi)^{d/2}} (\psi(x) + \psi(y)) \alpha_0 \left( \frac{x-y}{\hbar} \right) - \alpha_\Delta(x, y) \right|^2 \, dx \, dy$$

For $\beta^{-1} = T = T_c(1 - \hbar^2 D)$, the terms in the first two lines yield $\hbar^{4-d} \mathcal{E}^{GL}(\psi) + O(\hbar^{6-d})$ for $\psi \in H^2(C)$. The last line can be controlled by the $H^1(C)$ norm of the operator, yielding also an error term $O(\hbar^{6-d})$. 
Ideas in the Lower Bound

The key is to show that if $\Gamma$ is an approximate minimizer, then $\Gamma \approx [1 + e^{\beta H \Delta}]^{-1}$ for suitable $\psi$ (approximately) minimizing $\mathcal{E}^{\text{GL}}$.

**Step 1.** For any $\Gamma$ with $F^{\text{BCS}}(\Gamma) \leq F^{\text{BCS}}(\Gamma_0)$, the corresponding $\alpha$ satisfies

$$\alpha = \frac{\hbar}{2} \left( \psi(x) \hat{\alpha}_0(-i\hbar \nabla) + \hat{\alpha}_0(-i\hbar \nabla) \psi(x) \right) + \sigma$$

for some $\psi$ with $H^1(C)$ norm bounded independently of $\hbar$, and with $\|\sigma\|_{H^1} \leq O(h^{2-d/2})$.

**Step 2.** With $\psi$ as above, we compute

$$F^{\text{BCS}}(\Gamma) - F^{\text{BCS}}(\Gamma_0) = -\frac{T}{2} \text{Tr} \left[ \ln(1 + e^{-\beta H \Delta}) - \ln(1 + e^{-\beta H_0}) \right]$$

$$- h^{2-2d} \int_{C \times \mathbb{R}^d} V(h^{-1}(x-y)) \frac{1}{4} |\psi(x) + \psi(y)|^2 |\alpha_0(h^{-1}(x-y))|^2 \frac{dx \, dy}{(2\pi)^d}$$

$$+ T \mathcal{H}(\Gamma, \Gamma_\Delta) + \int_{C \times \mathbb{R}^d} V(h^{-1}(x-y)) |\sigma(x,y)|^2 \, dx \, dy,$$

where $\mathcal{H}$ denotes the relative entropy.
**Relative Entropy**

For general $\Gamma$ and $\Gamma_\Delta = [1 + e^{\beta H_\Delta}]^{-1}$, it is true that

$$\mathcal{H}(\Gamma, \Gamma_\Delta) = \text{Tr} \Gamma (\ln \Gamma - \ln \Gamma_\Delta) \geq \text{Tr} \left[ \frac{\beta H_\Delta}{\tanh \frac{1}{2} \beta H_\Delta} (\Gamma - \Gamma_\Delta)^2 \right]$$

Since $x \mapsto \sqrt{x}/\tanh \sqrt{x}$ is an operator monotone function, we can further bound

$$\frac{H_\Delta}{\tanh \frac{1}{2} \beta H_\Delta} \geq (1 - O(h)) \frac{H_0}{\tanh \frac{1}{2} \beta H_0} \geq (1 - O(h)) K_T(-i\hbar \nabla) \otimes \mathbb{I}_{\mathbb{C}^2}$$

Recall that, by definition, $K_{T_c}(-i\nabla) + V(x) \geq 0$, and hence $K_T(-i\nabla) + V(x) \geq -O(h^2)$. Moreover, $\alpha - \alpha_\Delta \approx \sigma$. This allows to get a lower bound on

$$TH(\Gamma, \Gamma_\Delta) + \int_{\mathbb{C} \times \mathbb{R}^d} V(h^{-1}(x - y))|\sigma(x, y)|^2 dx dy$$

that is $o(h^{4-d})$. 
Conclusion

- Rigorous derivation of Ginzburg-Landau theory, starting from the BCS model.
- For weak external fields and close to the critical temperature, GL arises as an effective theory on the macroscopic scale.
- The relevant scaling limit is semiclassical in nature.

Some open problems:

- Treat physical boundary conditions
- Treat self-consistent magnetic fields
- Derive BCS theory from many-body quantum mechanics
THANK YOU FOR YOUR ATTENTION!