A new uniqueness theorem for k-graph C*-algebras

Sarah Reznikoff

joint work with Jonathan H. Brown and Gabriel Nagy
Kansas State University

COSy 2013
Fields Institute
**Cuntz Algebra (1977):** $\mathcal{O}_n$, generated by $n$ partial isometries $S_i$ satisfying $\forall i, \ S_i^* S_i = \sum_{j=1}^{n} S_j S_j^*$.

**Cuntz-Krieger Algebras (1980):** $\mathcal{O}_A$, generated by partial isometries $S_1, \ldots, S_n$, with relations $S_i^* S_i = \sum_{j=1}^{n} A_{ij} S_j S_j^*$ for an $n \times n$ matrix $A$ over $\{0, 1\}$, i.e., the adjacency matrix of a finite directed graph with no multiple edges.

**Graph algebras:** generalization to arbitrary directed graphs.

**Generalizations and related constructions:** Exel crossed product algebras, Leavitt path algebras (Abrams, Ruiz, Tomforde), topological graph algebras (Katsura), Ruelle algebras (Putnam, Spielberg), Exel-Laca algebras, ultragraphs (Tomforde), Cuntz-Pimsner algebras, higher-rank Cuntz-Krieger algebras (Robertson-Steger), etc.

**Sarah Reznikoff**

A new uniqueness theorem for k-graph C*-algebras
k-graph algebras (Kumjian and Pask, 2000)

- developed to generalize graph algebras and higher-rank Cuntz-Krieger algebras,
- whether simple, purely infinite, or AF can be determined from properties of the graph (Kumjian-Pask, Evans-Sims),
- can be described from a k-colored directed graph—a "skeleton"—along with a collection of "commuting squares" (Fowler, Sims, Hazlewood, Raeburn, Webster),
- are groupoid C*-algebras,
- include examples of algebras that are simple but neither AF nor purely infinite, and hence not graph algebras (Pask-Raeburn-Rørdam-Sims),
- include examples that can be constructed from shift spaces (Pask-Raeburn-Weaver),
- can be used to construct any Kirchberg algebra (Spielberg).
Let $k \in \mathbb{N}^+$. We regard $\mathbb{N}^k$ as a category with a single object, 0, and with composition of morphisms given by addition.

A $k$-graph is a countable category $\Lambda$ along with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorization property:

For all $\lambda \in \Lambda$, and $m, n \in \mathbb{N}^k$, if $d(\lambda) = m + n$ then there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu \nu$.

- Denote the range and source maps $r, s : \Lambda \to \Lambda$.
- Refer to the objects of $\Lambda$ as vertices and the morphisms of $\Lambda$ as paths.
- Unique factorization implies that $d(\lambda) = 0$ iff $\lambda$ a vertex.
Illustration of unique factorization in $k = 2$ case.

Let $\lambda \in \Lambda$, with $d(\lambda) = (10, 8)$. Then $\lambda = \mu \nu$, with $d(\mu) = (4, 4)$ and $d(\nu) = (6, 4)$. The diagram shows the relationships $r(\lambda)$, $s(\lambda)$, $r(\mu)$, $s(\mu)$, $r(\nu)$, and $s(\nu)$.
1. The set $E^*$, where $(E^0, E^1, r, s)$ is a directed graph. Set $d(\lambda) = d$ iff $\lambda$ has length $d$.

2. Let $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$ with composition $(m, r)(r, n) = (m, n)$ and degree map $d(m, n) = n - m$. 

Sarah Reznikoff

A new uniqueness theorem for k-graph C*-algebras
3. We can define a 2-graph from the directed colored graph $E = (E^0, E^1, r, s)$ with color map $c : E^1 \to \{1, 2\}$ as follows.

Endow $E^*$ with the degree functor given by

$$d(e_1 e_2 \ldots e_n) = (m_1, m_2), \text{ where } m_i = |c^{-1}(i)|.$$

Since $(0, 1) + (1, 0) = (1, 0) + (0, 1)$ and the only paths of degrees $(1, 0)$ and $(0, 1)$ are, respectively, $e$ and $f$, to define a 2-graph from $E^*$ we must declare $ef = fe$. In fact, any two paths of equal degree must be equal.

The 2-graph we obtain is the semigroup $\mathbb{N}^2$ with degree map the identity.
Notation:

- For \( n \in \mathbb{N}^k \), we denote \( \Lambda^n = \{ \lambda \in \Lambda \mid d(\lambda) = n \} \).
- For \( v \in \Lambda^0 \) denote \( v\Lambda^n = \{ \lambda \in \Lambda^n \mid r(\lambda) = v \} \).

A \( k \)-graph \( \Lambda \) is **row-finite** and has **no sources** if

\[
\forall v \in \Lambda^0, \forall n \in \mathbb{N}^k, \ 0 < |v\Lambda^n| < \infty.
\]

Assume all \( k \)-graphs are row-finite and have no sources.
A Cuntz-Krieger $\Lambda$-family in a $C^*$-algebra $A$ is a set $\{T_\lambda, \lambda \in \Lambda\}$ of partial isometries in $A$ satisfying

(i) $\{T_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
(ii) $T_{\lambda \mu} = T_\lambda T_\mu$ for all $\lambda, \mu \in \Lambda$ s.t. $s(\lambda) = r(\mu)$,
(iii) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$, and
(iv) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, $T_v = \sum_{\lambda \in v \Lambda^n} T_\lambda T_\lambda^*$.

For $\lambda \in \Lambda$, denote $Q_\lambda := T_\lambda T_\lambda^*$.

$C^*(\Lambda)$ will denote the $C^*$-algebra generated by a universal Cuntz-Krieger $\Lambda$-family, $(S_\lambda, \lambda \in \Lambda)$, with $P_\lambda = S_\lambda S_\lambda^*$.
Q: When is a *-homomorphism $\Phi : C^*(\Lambda) \to A$ injective?

Necessary: $\Phi$ is nondegenerate, i.e., it is injective on the diagonal subalgebra $\mathcal{D} := C^*\left(\{P_\mu \mid \mu \in \Lambda\}\right)$.

Our new uniqueness theorem proves the sufficiency of injectivity on a (usually) larger subalgebra, $\mathcal{M} \supseteq \mathcal{D}$, and generalizes our theorem for directed graphs, where $\mathcal{M}$ is called the Abelian Core of $C^*(\Lambda)$.

Gauge Actions

The universal C*-algebra of a k-graph Λ has a gauge action \( \alpha : \mathbb{T}^k \to \text{Aut} \, C^*(\Lambda) \) given by

\[
\alpha_t(S_\lambda) = t^{d(\lambda)} S_\lambda = t_1^{d_1} t_2^{d_2} \ldots t_k^{d_k} S_\lambda,
\]

where \( t = (t_1, t_2, \ldots t_k) \) and \( d(\lambda) = (d_1, d_2, \ldots d_k) \).

**Gauge-Invariant Uniqueness Theorem** (Kumjian-Pask): If \( \Phi : C^*(\Lambda) \to A \) is a nondegenerate *-representation and intertwines a gauge action \( \beta : \mathbb{T}^k \to \text{Aut}(A) \) with \( \alpha \), then \( \Phi \) is injective.
Recall $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k | m \leq n\}$, with degree map $d(m, n) = n - m$ and composition $(m, n)(n, r) = (m, r)$.

The **infinite path space** $\Lambda^\infty$ is the set of all degree-preserving covariant functors $x : \Omega_k \to \Lambda$.

$x \in \Lambda^\infty$

$r(x)$

$s(\alpha)$

$r(\alpha)$

$\alpha = x((2, 4), (6, 6)) \in \Lambda^{(4,2)}$
For $\alpha \in \Lambda$ and $y \in \Lambda^\infty$, if $s(\alpha) = r(y)$ then $\alpha y$ is the unique $x \in \Lambda^\infty$ s.t. $x(0, N) = \alpha y(d(\alpha), N)$ for all $N \geq d(\alpha)$.

Using the topology generated by the cylinder sets

$$Z(\alpha) = \{ x \in \Lambda^\infty \mid x(0, d(\alpha)) = \alpha \} = \{ x \in \Lambda^\infty \mid \exists y \in \Lambda^\infty \text{ s.t. } x = \alpha y \},$$

$\Lambda^\infty$ is a locally compact Hausdorff space.
The shift map: For $x \in \Lambda^\infty$ and $N \in \mathbb{N}^k$, $\sigma^N(x)$ is defined to be the element of $\Lambda^\infty$ given by $\sigma^N(x)(m, n) = x(m + N, n + N)$.

$x \in \Lambda^\infty$ is *eventually periodic* if there is an $N \in \mathbb{N}^k$ and an $p \in \mathbb{Z}^k$ such that $\sigma^N(x) = \sigma^{N+p}(x)$; otherwise $x$ is *aperiodic*.

$x \in \Lambda^\infty$

$N = (1, 2)$

$p = (4, -1)$
**Theorem** (Kumjian-Pask) If $\Lambda$ satisfies

(A) for every $v \in \Lambda^0$ there is an aperiodic path $x \in v\Lambda^\infty$,

then any nondegenerate representation of $C^*(\Lambda)$ is injective.

**Theorem** (Raeburn, Sims, Yeend)
If $\Lambda$ satisfies

(B) For each $v \in \Lambda^0$ there is an $x \in v\Lambda^\infty$ s.t.

$$\forall \alpha, \beta \in \Lambda \quad (\alpha \neq \beta \Rightarrow \alpha x \neq \beta x)$$

then any nondegenerate representation of $C^*(\Lambda)$ is injective.

**Remarks:**
- When $\Lambda$ has no sources, (A) $\Rightarrow$ (B).
- (B) $\Rightarrow$ (A) holds for 1-graphs.
The super-normal subalgebra

Observation: $C^*(\Lambda) = \overline{\text{span}}\{S_\mu S_\nu^* | \mu, \nu \in \Lambda, \ s(\mu) = s(\nu)\}$. Recall $P_\alpha := S_\alpha S_\alpha^*$.

**Defn.** We call the element $S_\alpha S_\beta^*$ super-normal if it is normal and commutes with $\mathcal{D} := C^*(\{P_\mu\})$.

**Prop.** The following are equivalent for $\alpha \neq \beta$.

(i) $S_\alpha S_\beta^*$ is super-normal.

(ii) For all $\gamma \in \Lambda$, $P_{\alpha \gamma} = P_{\beta \gamma}$.

(iii) For all $\gamma \in s(\alpha)\Lambda$, the pair $(\alpha \gamma, \beta \gamma)$ is a generalized cycle without entry, in the sense of Evans and Sims.
**Example** ($k = 1$): Suppose $\lambda$ is a cycle without entry, $r(\lambda) = s(\alpha)$, and $\beta = \lambda \circ \alpha$. Then it is easy to verify that for all $\gamma \in \Lambda$, $P_{\alpha \gamma} = P_{\beta \gamma}$, so $S_{\alpha} S_{\beta}^*$ is super-normal.

On the other hand:

**Fact:** If $s(\alpha) = s(\beta)$ but $\alpha \neq \beta$, and there exists an aperiodic $x \in s(\alpha) \Lambda^{\infty}$, then $S_{\alpha} S_{\beta}^*$ is not super-normal.

Therefore, if $\Lambda$ satisfies Condition (A) then the only super-normal generators are the projections $P_{\mu} = S_{\mu} S_{\mu}^*$.
Let $\mathcal{M} = C^* \{ S_\alpha S_\beta^* \text{ super-normal}\}$.

**Theorem** (Brown-Nagy-R, 2013)
For a representation $\Phi : C^*(\Lambda) \to B$, TFAE:

(i) $\Phi$ is injective

(ii) $\Phi$ is injective on $\mathcal{M}$.

**Rmk:** By the observation on the previous page, if $\Lambda$ satisfies Condition (A) then $\mathcal{M} = \mathcal{D} := C^*\{P_\mu\}$.

The proof involves examining a representation of $C^*(\Lambda)$ in $B(\ell^2(X))$, for $X \subset \Lambda^\infty$ the set of “regular paths” of $\Lambda$. 
For $\alpha, \beta \in \Lambda$, let $F_{\alpha, \beta} := \{x \in \Lambda^\infty \mid \exists y \in \Lambda^\infty x = \alpha y = \beta y\}$.

Facts:

- $x \in F_{\alpha, \beta}$ is eventually periodic of period $p = d(\beta) - d(\alpha)$.
- Any eventually periodic $x$ is in some $F_{\alpha, \beta}$.
- $F_{\alpha, \beta}$ is closed, and if $\alpha = \beta$, then $F_{\alpha, \beta} = \mathcal{Z}(\alpha)$. 
The regular paths are the elements of

\[ X := \Lambda^\infty \setminus \bigcup_{\alpha, \beta \in \Lambda} \partial F_{\alpha, \beta}. \]

- \( X \) is dense in \( \Lambda^\infty \) (uses Baire Category).
- \( X \) is closed under the shift.
- When \( k = 1 \),
  \( X = \{ \text{infinite “essentially aperiodic” paths} \} \).

Aperiodic paths are essentially aperiodic.

If \( \lambda \) cycle with no entry, \( \alpha \in \Lambda, r(\lambda) = s(\alpha) \), then
\( x = \alpha \lambda^\infty \) is essentially aperiodic.
There is a Cuntz-Krieger family $(T_\alpha, \alpha \in \Lambda)$ in $B(\ell^2(X))$, given by

$$T_\alpha \delta_x = \begin{cases} 
\delta_{\alpha x} & \text{if } x \in s(\alpha)\Lambda^n \\
0 & \text{otherwise.}
\end{cases}$$

We define the *aperiodic representation*:

$$\pi_{ap} : \mathcal{C}^*(\Lambda) \to B(\ell^2(X))$$

$$S_\lambda \mapsto T_\lambda$$

We first prove that for representations of $\pi_{ap}(\mathcal{C}^*(\Lambda))$ injectivity on $\pi_{ap}(\mathcal{M})$ lifts.
Abstract Uniqueness Theorem (Brown-Nagy-R, 2013)
Let $A$ be a $C^*$-algebra and $M \subset A$ an abelian $C^*$-subalgebra. Suppose there is a set $S$ of pure states on $M$ satisfying

(i) each $\psi \in S$ extends uniquely to a state $\tilde{\psi}$ on $A$, and

(ii) the collection $\tilde{S} := \{\tilde{\psi} \mid \psi \in S\}$ is “jointly faithful” on $A$.

Then a $\ast$-homomorphism $\Phi : A \to B$ is injective iff $\Phi|_M$ is injective. Moreover, $M'$ is a masa in $A$.

**Corollary**
A $\ast$-representation $\Phi : \pi_{ap}(C^*(\Lambda)) \to B$ is injective iff it is injective on $\pi_{ap}(M)$.

**Proof:** The hypotheses of the Abstract Uniqueness Theorem hold with $S$ a set of “evaluation states”. (See extra slides after biblio. for proof sketches.)
To handle representations of $C^*(\Lambda)$:
Define the “twisted aperiodic representation”

$$\Psi_{ap} : C^*(\Lambda) \to B(\ell^2(X \times \mathbb{Z}^k)).$$

Now the gauge invariance theorem applies.
Adapt the previous argument to $\Psi_{ap}(C^*(\Lambda))$. Pull back the jointly faithful set of uniquely extending states to $C^*(\Lambda)$ to prove:

**Theorem** (Brown-Nagy-R, 2013)
For a representation $\Phi : C^*(\Lambda) \to B$, TFAE:

(i) $\Phi$ is injective.

(ii) $\Phi$ is injective on $C^*(\mathcal{M})$. 
(Renault, ‘80) A C*-subalgebra $B \subseteq A$ is **Cartan** if

- $B$ is a masa in $A$,
- $\exists$ a faithful conditional expectation $A \to B$,
- The normalizer of $B$ in $A$ generates $A$, and
- $B$ contains an approximate unit of $A$.

**Theorem** (Nagy-R, 2011)

If $\Lambda$ is a 1-graph then $\mathcal{M} \subseteq C^*(\Lambda)$ is Cartan.

**Defn.** $B \subseteq A$ has the Unique Extension Property (UEP) if every pure state on $B$ extends uniquely to a pure state on $A$.

- A Cartan C*-subalgebra with the UEP is a **C*-Diagonal**.
- For $k = 1$, $\mathcal{M}$ is a **pseudo-diagonal**: densely many pure states extend uniquely and there is a faithful conditional exp.
- For arbitrary $k$, $\mathcal{M}'$ is a MASA. Is it a pseudo-diagonal?


Thank you!
Sketch of corollary proof:

Let $A = \pi_{ap}(C^*(\Lambda))$, $M = \pi_{ap}(\mathcal{M})$, and $D = \pi_{ap}(\mathcal{D})$.

- Why $M$ is abelian: Note that if $T \in D'$ then $T$ commutes with all $p_x := \text{sot-lim}_{n \to \infty} Q_x(0,n)$ so $T \in \ell^\infty(X)$. Thus $D'$ is abelian, and $M \subseteq D'$ by definition.

- The states in $S$: For each $x \in X$ define $\text{ev}_x^D(Q_\alpha) = \chi_{Z(\alpha)}(x)$. Let $\phi$ be an extension of $\text{ev}_x^D$ to $A$. We show that $\phi(T_\alpha T_\beta^*)$ depends only on $x$, $\alpha$, and $\beta$. To do this, we extend $\alpha$ and $\beta$ to $\mu$ and $\nu$ with $T_\nu = T_\mu$. Denote the unique extension $\phi_x$ and let $S = \{ \phi_x|_M \mid x \in X \}$.

- Why the extensions $\phi_x$ are jointly faithful on $A$: Easy to see that $\phi_x(T) = \langle T \delta_x, \delta_x \rangle$ and so if $T = (T^{1/2})^2$ and $\phi_x(T) = 0$ for all $x$ then $T^{1/2} = 0$ too.
Ideas in proof of Abstract Uniqueness Theorem:

We are assuming the states \( \psi \in S \) on \( M \) extends uniquely to states \( \tilde{\psi} \in \tilde{S} \) on \( A \), and the collection of the extensions is jointly faithful on \( A \).

- If \( \ker \phi|_M \subseteq \ker \psi \) then \( \ker \phi \subseteq \ker \pi_\psi \) (the GNS representation associated with \( \psi \)).
- If \( \tilde{S} \) is jointly faithful then \( \bigcap_{\psi \in S} \ker \pi_\psi = \{0\} \).
The conditional expectation when $k = 1$:

For $x \in X$, let $p_x = \lim_{n \to \infty} Q_{x(0,n)} \in B(\ell^2(X))$.

$\triangleright$ $p_x$ is the projection onto $\text{span}\{\delta_{x,m} \mid m \in \mathbb{Z}^k\}$

$\triangleright \phi_x(T_{\alpha} T_{\beta}^*) p_x = p_x T_{\alpha} T_{\beta}^* p_x$.

Define

$$E_{ap} : B(\ell^2(X)) \to \{p_x \mid x \in X\}'$$

$$A \mapsto \sum_{x \in X} p_x A p_x$$

$E_{ap}$ is a faithful conditional expectation; moreover $\Psi_{ap}$ intertwines it with a faithful conditional expectation

$$E_{\Lambda} : C^*(\Lambda) \to \mathcal{M}.$$