Geometric classification of graph $C^*$-algebras

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Program

1. Graphs & moves
2. Classification
3. Finite graphs
Outline

1. Graphs & moves
2. Classification
3. Finite graphs
Graph algebras

Any countable graph $E = (E^0, E^1)$ defines a $C^*$-algebra $C^*(E)$ given as a universal $C^*$-algebra by projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ subject to the Cuntz-Krieger relations:

1. $p_v p_w = 0$ when $v \neq w$
2. $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
3. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e) = v} s_e s_e^*$ for every $v$ with $0 < |\{e | s(e) = v\}| < \infty$. 
There is a huge body of knowledge about graph algebras. Of prime importance here is

**Observation**

\[ \gamma_z(p_v) = p_v \quad \gamma_z(s_e) = zs_e \]

induces a **gauge action** \( \mathbb{T} \mapsto \text{Aut}(C^*(E)) \)

**Theorem**

*Gauge invariant ideals are induced by hereditary and saturated sets of vertices \( V \):*

- \( s(e) \in V \implies r(e) \in V \)
- \( r(s^{-1}(v)) \subseteq V \implies [v \in V \text{ or } v \text{ is singular (\(\circ\))}] \)

*and when there are no breaking vertices, all arise this way.*
The gauge simple case

Theorem

If a graph $C^*$-algebra has no non-trivial gauge invariant ideals, it is either

1. an AF algebra;
2. a Kirchberg algebra; or
3. $C(\mathbb{T}) \otimes K(H)$ for some Hilbert space $H$.

It is easy to tell from the graph which case occurs. The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.
The unital case

**Observation**

\( C^*(E) \) is unital \( \iff \) \( E_0 \) is finite

In this case we get a finite presentation, e.g.

\[
A_E = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\infty & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix} \quad A_E^\bullet = \begin{bmatrix}
0 & 0 \\
1 & 1 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix} \quad A_E^\circ = \begin{bmatrix}
0 & 0 \\
\infty & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

for

\[
\begin{array}{c}
\circ \\
\longrightarrow \\
\bullet \\
\circ \\
\end{array}
\overset{\circ} \rightarrow \overset{\bullet} \rightarrow \overset{\bullet} \rightarrow \overset{\circ} \rightarrow
\]

\[
\begin{array}{c}
\circ \\
\overset{\bullet} \rightarrow \overset{\bullet} \rightarrow \overset{\bullet} \rightarrow \overset{\circ} \rightarrow
\end{array}
\]
Moves

**Move (S)**
Remove a regular source, as

![Diagram for Move (S)](image)

**Move (R)**
Reduce a configuration with a transitional regular vertex, as

![Diagram for Move (R)](image)

or

![Alternative Diagram for Move (R)](image)
Move (I)
Insplit at regular vertex

\[\begin{align*}
\bullet & \rightarrow \star \\
\star & \rightarrow \bullet \\
\bullet & \rightarrow \star \\
\bullet & \rightarrow \bullet
\end{align*}\]
Move (O)

Outsplit at any vertex (at most one group of edges infinite)
Definition

\( E \sim_{C^*} F \) when \( C^*(E) \otimes K \simeq C^*(F) \otimes K \)

Definition

\( E \sim_m F \) when there is a finite sequence of moves of type

\((S),(R),(O),(I),\)

and their inverses, leading from \( E \) to \( F \).
Proposition

\[ E \sim_m F \iff E \sim_{C^*} F \]

Our goal will be to try to reverse this implication.
Observation
\[ \dim C^*(E) < \infty \text{ precisely when } E \text{ is a finite graph with no cycles} \]

Let \( \#_{\text{sink}}(E) \) denote the number of sinks in the graph \( E \).

Theorem
Assume that \( E, F \) are both finite graphs with no cycles. The following are equivalent

1. \( E \sim_{C^*} F \)
2. \( E \sim_m F \) via moves (O) and (R)
3. \( \#_{\text{sink}}(E) = \#_{\text{sink}}(F) \)
Amplified graphs

**Move (T)**

Emit infinitely to any vertex reachable by a path starting with an edge with infinitely many parallels:

```
○   →    ●   →    ●   ⇝   ○   ↔   ●   →    ●   →    ●
```

**Lemma (E-Ruiz-Sørensen)**

*Move (T) is generated by moves (R), (I), and (O).*
Theorem (E-Ruiz-Sørensen)

When $E$ and $F$ both have finitely many vertices and both have the property that if there is an edge between two vertices, there are infinitely many (i.e., they are amplified), then

$$E \sim_{C^*} F \iff E \sim_m F$$

through move (T) only.
Let $E$ be a finite graph with no sinks or sources. Then

$$X_E = \{(e_n) \in (E^1)^\mathbb{Z} \mid r(e_n) = s(e_{n+1})\}$$

is a shift space; in fact an SFT.

We say that two shift spaces $X$ and $Y$ are flow equivalent and write $X \sim_{FE} Y$ when their suspension flows

$$SX = X \otimes \mathbb{R}/\langle (x, t) \sim (\sigma(x), t + 1) \rangle$$

are homeomorphic in a way preserving the direction of the flow lines. The relevance of this notion comes from

**Theorem (Parry-Sullivan)**

*When $E$, $F$ are both finite graphs with no sinks or sources, then*

$$X_E \sim_{FE} X_F \iff E \sim_m F$$
Theorem (Franks)

Let $E$ and $F$ be strongly connected finite graphs, not a single cycle. The following are equivalent

1. $\mathbb{Z}^n / \text{im}(I - A_E) \simeq \mathbb{Z}^m / \text{im}(I - A_F)$ and $\det(I - A_E) = \det(I - A_F)$
2. $X_E \sim_{FE} X_F$
Consider the two graphs $E, F$ given by

We have

$$\mathbb{Z}/\text{im}(I - A_E) = 0 = \mathbb{Z}^2/\text{im}(I - A_F)$$

but

$$\det(I - A_E) = -1 \neq 1 = \det(I - A_F).$$

**Theorem (Rørdam)**

$$C^*(E) \otimes K \simeq C^*(F) \otimes K$$

Hence $E \sim_{C^*} F$ but $E \not\sim_m F$. 
**Move (C)**

“Cuntz splice” on a vertex supporting two cycles

![Diagram](image)

**Theorem (Rørdam, Cuntz)**

Let $E, F$ be strongly connected finite graphs, not a single cycle. When $E \sim_{C^*} F$, then $\mathbb{Z}^n/(I - A_E)\mathbb{Z}^n \simeq \mathbb{Z}^m/(I - A_F)\mathbb{Z}^m$. When $\mathbb{Z}^n/(I - A_E)\mathbb{Z}^n \simeq \mathbb{Z}^m/(I - A_F)\mathbb{Z}^m$, then $E \sim_M F$ through moves (R), (I), (O), and (C).

**Definition**

$E \sim_M F$ when there is a finite sequence of moves of type

$$(S), (R), (O), (I), (C)$$

and their inverses, leading from $E$ to $F$. 
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Filtered $K$-theory

**Definition**

Let $\mathfrak{A}$ be a $C^*$-algebra having only finitely many ideals. The collection of all sequences

$$
\begin{align*}
K_0(J/I) &\longrightarrow K_0(K/I) \longrightarrow K_0(K/J) \\
K_1(K/J) &\leftarrow K_1(K/I) \leftarrow K_1(J/I)
\end{align*}
$$

with $I \triangleleft J \triangleleft K \triangleleft \mathfrak{A}$ gauge invariant ideals is called the *filtered $K$-theory* of $\mathfrak{A}$ and denoted $FK(\mathfrak{A})$. Equipping all $K_0$-groups with order we arrive at the *ordered,filtered $K$-theory* $FK_+(\mathfrak{A})$. 
Question

Suppose \( C[X] \) is a family of \( C^* \)-algebras with real rank zero and primitive ideal space \( X \), so that it is known that \((K_*(-), K_*(-)_{+})\) is a complete invariant for all simple subquotients of \( A \in C \).

When can we conclude that \( FK^+(-) \) is a complete invariant for the \( A \)'s themselves?

Theorem (Elliott, Kirchberg-Phillips)

\[
K_*(C^*(E)) \cong K_*(C^*(F)) \iff C^*(E) \otimes K \cong C^*(F) \otimes K
\]
**Status quo**

$FK^+(-)$ is known to be a complete invariant for graph $C^*$-algebras over $X$ when

- $|X| = 2$ [E-Tomforde]
- $|X| = 3$ and all $K$-groups are finitely generated [E-Restorff-Ruiz]

and in roughly 2/3 of the possible cases with $|X| = 4$ [Arklint-Bentmann-E-Katsura-Köhler-Restorff-Ruiz]. No counterexamples are known.
Theorem (Sørensen)

Let $E$ and $F$ be graphs so that $C^*(E)$ and $C^*(F)$ are unital and gauge simple. The following are equivalent

1. $K_*(C^*(E)) \simeq K_*(C^*(F))$
2. $E \sim_M F$
3. $E \sim_{C^*} F$
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Gauge filtered $K$-theory

**Definition**

Let $(\mathcal{A}, \gamma)$ be a $C^*$-algebra with a gauge action, having only finitely many gauge-invariant ideals. The collection of all sequences

$$
\begin{align*}
K_0(\mathcal{I}/\mathcal{J}) &\longrightarrow K_0(\mathcal{K}/\mathcal{I}) \longrightarrow K_0(\mathcal{K}/\mathcal{J}) \\
K_1(\mathcal{K}/\mathcal{J}) &\leftarrow K_1(\mathcal{K}/\mathcal{I}) \leftarrow K_1(\mathcal{I}/\mathcal{J})
\end{align*}
$$

with $\mathcal{I} \triangleleft \mathcal{J} \triangleleft \mathcal{K} \triangleleft \mathcal{A}$ gauge invariant ideals is called the *gauge filtered $K$-theory* of $\mathcal{A}$ and denoted $FK_{\gamma}(\mathcal{A})$. Equipping all $K_0$-groups with order we arrive at the *ordered, gauge filtered $K$-theory* $FK_{\gamma,+}(\mathcal{A})$. 
The **reduced** ordered, gauge filtered $K$-theory $FK_{\gamma,+,\text{red}}(\mathfrak{A})$ consists of

$$
\begin{align*}
K_0(\mathfrak{J}) & \rightarrow K_0(\mathfrak{K}) \rightarrow K_0(\mathfrak{K}/\mathfrak{J}) \\
& \uparrow \\
K_1(\mathfrak{K}/\mathfrak{J}) &
\end{align*}
$$

with $\mathfrak{J}$ a maximal gauge invariant ideal inside a gauge prime ideal $\mathfrak{K}$, along with

$$K_0(\mathfrak{J}_n) \rightarrow K_0(\mathfrak{J})$$

whenever $\mathfrak{J}, \mathfrak{J}_n$ are gauge prime with $\mathfrak{J} = \bigcup \mathfrak{J}_n$. 
Definition

The **signature** of \( \mathcal{A} \) with finitely many gauge invariant ideals is a map

\[
\tau : \text{Prim}^\gamma(\mathcal{A}) \to \mathbb{Z}
\]

given by

\[
\tau(\mathcal{I}) = \begin{cases} 
-2 & \text{if } \mathcal{I}/\mathcal{I}_0 \text{ is not simple} \\
-1 & \text{if } \mathcal{I}/\mathcal{I}_0 \text{ is simple and AF} \\
\text{rank } K_0(\mathcal{I}/\mathcal{I}_0) - \text{rank } K_1(\mathcal{I}/\mathcal{I}_0) & \text{otherwise}
\end{cases}
\]

when \( \mathcal{I}_0 \) is the maximal gauge invariant proper ideal of \( \mathcal{I} \).
**Theorem (E-Ruiz-Sørensen)**

Let $E$ and $F$ be finite graphs. Then the following are equivalent

1. $E \sim_M F$
2. $E \sim_{C^*} F$
3. $FK_{\gamma,+}(C^*(E)) \simeq FK_{\gamma,+}(C^*(F))$
4. $\tau_E = \tau_F$ and $FK_{\gamma,+}^{\text{red}}(C^*(E)) \simeq FK_{\gamma,+}^{\text{red}}(C^*(F))$

Restorff proved $(2) \iff (3) \iff (4)$ for $E, F$ with no sinks and sources, and every vertex reaching a vertex supporting two cycles (condition (II)).