Triality over arbitrary fields and over $\mathbb{F}_1$

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Outline

- Some history
- Triality over arbitrary fields (Chernousov, Tignol, K., 2011)
- Triality over $\mathbb{F}_1$ (Tignol, K., 2012)
I. Some history

Wikipedia:

“There is a geometrical version of triality, analogous to duality in projective geometry.

... one finds a curious phenomenon involving 1, 2, and 4 dimensional subspaces of 8-dimensional space ...”
Geometric triality

▶ \((V, q)\) : Quadratic space of dimension 8 of maximal index.
\(U_i\) : Set of isotropic subspaces of \(V\) of dimension \(i, i \leq 4\).

▶ “Projective” terminology :
\(Q = \{q = 0\}\) defines a 6-dimensional quadric in \(\mathbb{P}^7\), the elements of \(U_i, i = 1, 2, 3, 4\), are **points, lines, planes and solids of** \(Q\).

▶ Two solids are of the **same kind** if their intersection is of even dimension. Two solids are of the same kind if and only if one can be transformed in the other by a rotation.
⇒ 2 kinds of solids !
Eduard Study

Grundlagen und Ziele der analytischen Kinematik, 1913

I The variety of solids of a fixed kind in $Q^6$ is isomorphic to a quadric $Q^6$.

II Any proposition in the geometry of $Q^6$ [about incidence relations] remains true if the concepts points, solids of one kind and solids of the other kind are cyclically permuted.
In other words, geometric triality is a geometric correspondence of order 3

\[
\text{Points} \rightarrow \text{Solids 1} \rightarrow \text{Solids 2} \rightarrow \text{Points}
\]

which is compatible with incidence relations.

In analogy to **geometric duality** which is a geometric correspondence

\[
\text{Points} \rightarrow \text{Hyperplanes}
\]

in projective space.

The word **triality** goes back to Élie Cartan: “On peut dire que le *principe de dualité* de la géométrie projective est remplacé ici par un *principe de trialité*”. 
The group $\text{PGO}_8^+$ admits a group of outer automorphisms isomorphic to $S_3$.

Outer automorphisms are related to “Cayley octaves”.

Outer automorphisms of order 3 will be called trialitarian automorphisms.
Cayley octaves or Octonions

- Octonions are a 8-dimensional algebra $\mathbb{O}$ with unit, norm $n$ and conjugation $x \mapsto \overline{x}$ such that
  \[ n(x) = x \cdot \overline{x} = \overline{x} \cdot x, \quad n(x \cdot y) = n(x)n(y). \]

- Cartan:
  Given $A \in \text{SO}(n)$ there exist $B, C \in \text{SO}(n)$ such that
  \[ C(x \cdot y) = Ax \cdot By. \]

  $\sigma : A \mapsto B, \quad \tau : A \mapsto C$ induce $\hat{\sigma}, \hat{\tau} \in \text{Aut}(\text{PGO}^+(n))$ such that
  \[ \hat{\sigma}^3 = 1, \quad \hat{\tau}^2 = 1, \quad \langle \hat{\sigma}, \hat{\tau} \rangle = S_3 \text{ in Aut}(\text{PGO}^+(n)). \]
The orthogonal projective group

- \( \text{PGO}(n) = \text{GO}(n)/F^\times, \)
  \[ \text{GO}(n) = \{ f \in \text{GL}(\mathbb{O}) \mid n(f(x)) = \mu(f)n(x) \}, \mu(f) \in F^\times. \]
- \( \text{PGO}^+(n) = \text{GO}^+ / F^\times, \) where \( \text{GO}^+(n) \) is the subgroup of \( \text{GO}(n) \) of direct similitudes (or projectively, of similitudes which respect the two kinds of solids).

**Notation:** \( f \in \text{GO}(n) \mapsto [f] \in \text{PGO}(n) \)
Octaves and geometric triality

Félix Vaney, Professeur au Collège cantonal, Lausanne, PhD-Student of É. Cartan, 1929:

I Solids are of the form

1. \( K_a = \{ x \in O \mid a \cdot x = 0 \} \) and 2. \( R_a = \{ x \in O \mid x \cdot a = 0 \} \).

II Geometric triality can be described as

\[ a \mapsto K_a \mapsto R_a \mapsto a. \]

for all \( a \in O \) with \( n(a) = 0 \).
A selection of later works

E. A. Weiss (1938,1939) : More (classical) projective geometry
É. Cartan (1938) : Leçons sur la théorie des spineurs
N. Kuiper (1950) : Complex algebraic geometry
H. Freudenthal (1951) : Local and global triality
C. Chevalley (1954) : The algebraic theory of spinors
J. Tits (1958) : Triality for loops
J. Tits (1959) : Classification of geometric trialities over arbitrary fields
F. van der Blij, T. A. Springer (1960) : Octaves and triality
T. A. Springer (1963) : Octonions, Jordan algebras and exceptional groups
N. Jacobson (1964) : Triality for Lie algebras over arbitrary fields.

Books (Porteous, Lounesto, [KMRT], Springer-Veldkamp).
II. Triality over arbitrary fields

with V. Chernousov and J-P. Tignol
Simple groups with trialitarian automorphisms

$G$ simple algebraic group with a trialitarian automorphism

$\Rightarrow$

$G$ of type $D_4$

**Reason** $D_4$ is the only Dynkin diagram with an automorphism of order 3

![Dynkin diagram](diagram)

**Theorem** $G$ of classical type $^{1,2}D_4$ with a trialitarian automorphism

$\Rightarrow G = \text{PGO}^+(n)$ or $G = \text{Spin}(n)$, $n$ a 3-Pfister form.
Aim

- Classify all trialitarian automorphisms of $\text{PGO}^+(n)$, up to conjugacy.
- Classify all geometric trialities up to isomorphism.

Method  Reduce to the (known) classification of a certain class of composition algebras.

Remark  Similar results for $\text{Spin}(n)$. 
M. Rost (≈1994)

There is a class of composition algebras well suited for triality, which Rost called **symmetric compositions**.
Symmetric compositions

A composition algebra is a quadratic space \((S, n)\) with a bilinear multiplication \(\star\) such that the norm of multiplicative :

\[
\text{n}(x \star y) = \text{n}(x) \star \text{n}(y)
\]

They exist only in dimension 1, 2, 4 and 8 (Hurwitz).

A symmetric composition satisfies

\[
x \star (y \star x) = (x \star y) \star x = \text{n}(x)y
\]

and

\[
b(x \star y, z) = b(x, y \star z).
\]

Remark For octonions the relations are

\[
\overline{x}(xy) = (yx)\overline{x} = \text{n}(x)y
\]

and

\[
b(xy, z) = b(x, z\overline{y}).
\]
Some history

Symmetric compositions existed already!

- **Petersson (1969)**: Einfach involutorische Algebren
  The product $x \star y = \bar{x} \bar{y}$ on an octonion algebra defines a symmetric composition ("para-octonions").

- **Okubo (1978)**: Pseudo-octonions algebras
  \[ S = M_3(F)^0, \ x \star y = \frac{yx - \omega xy}{1 - \omega} - \frac{1}{3} \text{tr}(xy), \ \text{Char} F \neq 3, \ \omega^3 = 1. \]

- **Faulkner (1988)**: Trace zero elements in cubic separable alternative algebras.

**Classification (Elduque-Myung, 1993)** Over fields of characteristic different from 3 8-dimensional symmetric compositions are either para-octonions or Okubo algebras attached to central simple algebras of degree 3.
Zorn matrices

The para-Zorn algebra

\[ Z = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \right| \alpha, \beta \in F, \ a, b \in F^3 \right\} \]

\[
\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \ast \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \beta \delta + a \cdot d & -\beta c - \gamma a - b \times d \\ -\delta b - \alpha d + a \times c & \alpha \gamma + b \cdot c \end{pmatrix},
\]

The Petersson twist

\[ x \ast_\theta y = \theta(x) \ast \theta^{-1}(y) \]

\[
\theta\left( \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \right) = \begin{pmatrix} \alpha & a^\varphi \\ b^\varphi & \beta \end{pmatrix}, \quad \varphi : (a_1, a_2, a_3) \mapsto (a_2, a_3, a_1)
\]

Theorem (Petersson, Elduque-Perez) Symmetric compositions are forms of the para-Zorn algebra and its Petersson twist.
A variation (Chernousov, Tignol, K., 2011)

$(S, n) : 3$-fold Pfister form ($\Leftrightarrow$ norm of an octonion algebra)

**Symmetric composition** : $\star : S \times S \to S$ such that

- $n(x \star y) = \lambda \cdot n(x)n(y)$, $\lambda \in F^\times$ (\(\lambda\) is the **multiplier** of $\star$)
- $b(x \star y, z) = b(x, y \star z)$

**Explanation**  This definition is more suited to deal with similitudes, $\lambda = 1$, “normalized symmetric composition”
Symmetric compositions and triality

Theorem

$(S, ⋆, n)$ a symmetric composition of dimension 8,

I Given $f \in \text{GO}^+(n)$, there exists $g, h \in \text{GO}^+(n)$, such that

$$f(x ⋆ y) = g(x) ⋆ h(y).$$

II the map $\rho_* : [f] \mapsto [g]$ is an outer automorphism of order 3 of $\text{PGO}^+(n)$ and $\rho_2^2[f] = [h]$.

Proof: With Clifford algebras, see [KMRT].

Remark: “Like” Cartan, but more symmetric!
More trialitarian automorphisms

There is a split exact sequence

\[ 1 \rightarrow \text{PGO}^+(n) \rightarrow \text{Aut} (\text{PGO}^+(n)) \rightarrow S_3 \rightarrow 1 \]

**Consequence**

\(\rho_*\) a fixed trialitarian automorphism of \(\text{PGO}^+(n)\)

\(\rho\) any trialitarian automorphism of \(\text{PGO}^+(n)\).

Then there exists \(f \in \text{GO}^+(n)\) such that

\[
\rho \text{ or } \rho^{-1} = \text{Int}([f]^{-1}) \circ \rho_* \quad \text{and} \quad f^{-1} \rho_*(f^{-1}) \rho_*^2(f^{-1}) = 1.
\]
Theorem (CKT, 2011) : The rule $\star \mapsto \rho_\star$ defines a bijection

Sym. comp. on $(S, n)$ up to scalars $\iff$ Trialit. aut. of $\text{PGO}^+(n)$

Proof of surjectivity

Given : $\rho$ a trialitarian automorphism.

1) Choose a fixed symmetric composition $\star$.

2) Take $f \in \text{GO}^+(n)$ such that $\rho$ or $\rho^{-1} = \text{Int}([f]^{-1}) \circ \rho_\star$ and $f^{-1}\rho_\star(f^{-1})\rho_\star^2(f^{-1}) = 1$ as above.

3) Pick $g \in \text{PGO}^+(n)$ such that $[g] = \rho_\star^2[f^{-1}]$.

Then $x \diamond y = f(x) \star g(y)$ is such that $\rho$ or $\rho^{-1} = \rho_\diamond$. 
Trialitarian automorphisms up to conjugacy

**Theorem** (Chernousov, Tignol, K., 2011):

\[
\text{Isomorphism classes of symmetric compositions with norm } n \iff \text{Conjugacy classes of trialitarian automorphisms of } \text{PGO}^+(n)
\]
Consequences

1. The classification of 8-dimensional symmetric compositions (Elduque-Myung, 1993) yields the classification of conjugacy classes of trialitarian automorphisms of groups $\text{PGO}^+(n)$.

2. Conversely one can first classify conjugacy classes of trialitarian automorphisms of groups $\text{PGO}^+(n)$ (Chernousov, Tignol, K., 201?) and deduce from it the classification of 8-dimensional symmetric compositions.
Symmetric compositions and geometric triality

Theorem
Given: $(S, \star, n)$ a 8-dimensional symmetric composition with hyperbolic norm.

Claim:
I All solids of one kind are of the form $x \star S$ and those of the other kind of the form $S \star y$, $x, y \in S$.

II The rule
$$\tau_\star : x \mapsto x \star S \mapsto S \star x \mapsto x$$
is a geometric triality.

III the rule $\star \mapsto \tau_\star$ defines a bijection

Sym. comp. on $(S, n)$ up to scal. $\Leftrightarrow$ Geom. trialit. on $\{n = 0\}$
Automorphisms of symmetric compositions

**Theorem:** \([ \text{PGO}^+(n)]^\rho \star = \text{Aut}(S, \star)\)

- \((S, \star)\) para-octonions \(\Rightarrow [ \text{PGO}^+(n)]^\rho \star \) of type \(G_2\).
- \((S, \star)\) Okubo, \text{Char } F \neq 3 \Rightarrow [ \text{PGO}^+(n)]^\rho \star \) of type \(A_2\).
- \((S, \star)\) Okubo, \text{Char } F = 3, is still mysterious!
Groups with triality of outer type $^{3,6}D_4$

“Outer types” are related with

- Semilinear trialities (in projective geometry)
- Generalized hexagons (incidence geometry, Tits, Schellekens, ...)
- Twisted compositions ($F_4$, Springer)
- Trialitarian algebras (KMRT)
III. Triality over $\mathbb{F}_1$

(with J-P. Tignol, 2012)
Sur les analogues algébriques des groupes semi-simples complexes, 1957

"Nous désignerons par $K = K_1$ le « corps de caractéristique 1 » formé du seul élément $1 = 0$ \(^{(19)}\). Il est naturel d’appeler espace projectif à $n$ dimensions sur $K$, un ensemble $\mathcal{P}_n$ of $n + 1$ points dont tous les sous-ensembles sont considérés comme des variétés linéaires {...}.

\(^{(19)}\) $K_1$ n’est généralement pas considéré comme un corps."
Vector spaces over $\mathbb{F}_1$

Since there is only one scalar, one has to work only with bases!

- $n$-dimensional vector space: $\mathcal{V} = \{x_1, \ldots, x_n, 0\}$

- $n-1$-dimensional projective space:
  $$\mathbb{P}(\mathcal{V}) = \langle \mathcal{V} \rangle = \{x_1, \ldots, x_n\}$$

  $\Rightarrow \text{Aut}(\mathcal{V}) = \text{Aut}(\langle \mathcal{V} \rangle) = GL_n(\mathbb{F}_1) = PGL_n(\mathbb{F}_1) = S_n.$

**Tits’ motivation**  There are algebraic (or geometric) objects whose automorphism groups are the simple algebraic groups. Tits wanted algebraic (or geometric) objects whose automorphism groups are the **Weyl groups** of these simple algebraic groups.
Quadratic spaces over $\mathbb{F}_1$

- A $2n$-dimensional quadratic space is a pair $Q = (V, \sim)$ where $V$ is a $2n$-dimensional vector space over $\mathbb{F}_1$ and $\sim : V \rightarrow V$ is a bijective self-map of order 2 such that $\sim 0 = 0$ and without other fixed points: $V = \{x_1, \ldots, x_n, y_1, \ldots y_n, 0\}$, $\tilde{x}_i = y_i$, $\tilde{y}_i = x_i$, $\tilde{0} = 0$.

- $\langle Q \rangle = Q \setminus \{0\}$ is the quadric associated to $Q$.

- $\langle Q \rangle$ is a double covering!

Example: $(V, q)$ “classical” hyperbolic space with hyperbolic basis

$$\{e_i, f_i, i \leq i \leq n \mid q(e_i) = q(f_i) = 0, b(e_i, f_j) = \delta_{ij}\}.$$ Set $\tilde{e}_i = f_i$, $\tilde{f}_i = e_i$. 
Let $Q = (\mathcal{V}, \tilde{\sim})$ be a $2n$-dimensional quadratic space over $\mathbb{F}_1$ and let $\mathcal{U}$ be a linear subspace of $\mathcal{V}$.

- $\mathcal{U}^\perp = \{ x \in \mathcal{V} \mid \tilde{x} \notin \mathcal{U} \} \sqcup \{ 0 \}$;
- $\mathcal{U}$ is isotropic if $\mathcal{U} \subset \mathcal{U}^\perp$ and maximal isotropic if $\mathcal{U} = \mathcal{U}^\perp$;
- $\mathcal{U}$ isotropic $\Rightarrow \dim \mathcal{U} \leq n$;
- Two kinds of maximal isotropic spaces: two maximal isotropic spaces $\mathcal{U}$ and $\mathcal{U}'$ are of the same kind if $\dim(\mathcal{U} \cap \mathcal{U}')$ has the same parity as $\frac{\dim \mathcal{V}}{2}$;
- $\mathcal{U}$ maximal isotropic $\iff \langle \mathcal{U} \rangle$ is a section of the double covering $\langle Q \rangle$;
Orthogonal groups over $\mathbb{F}_1$

$$\text{O}(Q) = \text{PGO}(\langle Q \rangle) = \text{PGO}_{2n}(\mathbb{F}_1) = S_2^n \rtimes S_n,$$

$$\text{O}^+(Q) = \text{PGO}^+(\langle Q \rangle) = \text{PGO}_{2n}^+(\mathbb{F}_1) = S_2^{n-1} \rtimes S_n$$
Trialitarian automorphisms of $\text{PGO}_8^+(\mathbb{F}_1)$

Known facts:

I. The Weyl group $S_2^3 \rtimes S_4$ of type $D_4$ (which is $\text{PGO}_8^+(\mathbb{F}_1)$) admits outer automorphisms of order 3.

II. If $\alpha$, $\beta$ are trialitarian automorphisms of $\text{PGO}_8^+(\mathbb{F}_1)$, then $\alpha \circ \beta^{-1}$ or $\alpha \circ \beta^{-2}$ is an inner automorphism.

Aim: Describe trialitarian automorphisms and geometric triality over $\mathbb{F}_1$ with symmetric compositions over $\mathbb{F}_1$!
Algebras over $\mathbb{F}_1$

A finite-dimensional algebra $(S, \star)$ over $\mathbb{F}_1$ is a finite-dimensional $\mathbb{F}_1$-vector space $S$ together with a map

$$\star: S \times S \rightarrow S, \quad (x, y) \mapsto x \star y,$$

called the multiplication, such that $0 \star x = x \star 0 = 0$ for all $x \in S$. 
Symmetric compositions over $\mathbb{F}_1$

A symmetric composition is a quadratic space $(S, \sim)$ with an algebra multiplication $\star$ satisfying the following properties for all $x, y \in S$:

(SC1) $\widetilde{x} \star \widetilde{y} = \check{x} \star \check{y}$.

(SC2) If $x, y \neq 0$, then
\[ x \star y = 0 \iff x \star \check{y} \neq 0 \iff \check{x} \star y \neq 0 \iff \check{x} \star \check{y} = 0. \]

(SC3) If $x \star y \neq 0$, then $(x \star y) \star \check{x} = y$ and $\check{y} \star (x \star y) = x$.

(SC4) If $x \star y = 0$, then $(x^{\perp} \star y) \star x = y \star (x \star y^{\perp}) = \{0\}$; i.e.,
\[ (u \star y) \star x = y \star (x \star v) = 0 \text{ for all } u \neq \check{x} \text{ and } v \neq \check{y}. \]
Maximal isotropic spaces = solids

Theorem

I The sets $x \star S$ and $S \star y$, $x, y \in S$ are solids of $\langle S \rangle$ of different kinds;

II Any solid is of the form $x \star S$ or $S \star y$.

III $\dim S = 2, 4$ or $8$.

Proof of III: $2^n \leq 4n$, so $n \leq 4$!
Examples in dimension 8

We use a “monomial” multiplication table for a “classical symmetric composition” and forget scalars!

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Symmetric compositions, trialitarian automorphisms and geometric triality over \( \mathbb{F}_1 \)

**Theorem (Tignol, K., 2012):** The rules

\[
\star \mapsto \rho_\star, \quad \rho_\star[f] = [g], \quad \text{if} \quad f(x \star y) = g(x) \star h(y)
\]

and

\[
\star \mapsto \tau_\star \quad \text{where} \quad \tau_\star : x \mapsto x \star S \mapsto S \star x \mapsto x
\]

define bijections

\[
\text{Trialit. aut. of } \text{PGO}_8^+(\mathbb{F}_1) \iff 8\text{-dim. sym. comp.} \iff \text{Geom. trialities}
\]
Back to geometric trialities

Let \( \langle Q \rangle \) be the quadric associated to an 8-dimensional quadratic space \( Q \) over \( \mathbb{F}_1 \).

▶ \( C = \{ \text{solids of } \langle Q \rangle \} \);

▶ The choice of a decomposition \( C = C_1 \sqcup C_2 \) into the two kinds of solids is an **orientation** of \( \langle Q \rangle \);
A geometric triality on $\langle Q \rangle$ is a pair $(\tau, \partial)$, where $\partial$ is an orientation $C = C_1 \sqcup C_2$ of $Z$ and $\tau$ is a map

$$\tau : Z \sqcup C_1 \sqcup C_2 \to Z \sqcup C_1 \sqcup C_2$$

with the following properties:

(GT1) $\tau$ commutes with the structure map $\sim : x \mapsto \tilde{x}$;

(GT2) $\tau$ preserves the incidence relations;

(GT3) $\tau(\langle Q \rangle) = C_1$, $\tau(C_1) = C_2$, and $\tau(C_2) = \langle Q \rangle$;

(GT4) $\tau^3 = I$.

The image of a line under $\tau$ is again a line!
Absolute points

An absolute point of a geometric triality \((\tau, \partial)\) is a point 
\(x \in \langle Q \rangle\) such that \(x \in \tau(x)\).

Theorem (Tignol, K.)

1) Suppose \((\tau, \partial)\) is a triality on \(\langle Q \rangle\) for which there exists an absolute point. Then the pair \((V, E)\) where \(V\) is the set of absolute points of \(\langle Q \rangle\) and \(E\) is the set of lines fixed under \(\tau\) is an hexagon:

\[(\text{absolute points, fixed lines}) = (V, E) = \]

Moreover, for every hexagon \((V, E)\) in \(\langle Q \rangle\) and any orientation \(\partial\) there is a unique geometric triality \((\tau, \partial)\) on \(\langle Q \rangle\) such that \(V\) is the set of absolute points of \(\tau\) and \(E\) is the set of fixed lines under \(\tau\).
2) Let \((\tau, \partial)\) be a geometric triality on \(\langle Q \rangle\) without absolute points. There are four hexagons \((V_1, E_1), \ldots, (V_4, E_4)\) with disjoint edge sets such that each edge set \(E_i\) is preserved under \(\tau\) and \(E_1 \sqcup E_2 \sqcup E_3 \sqcup E_4\) is the set of all lines in \(\langle Q \rangle\).

\[
\text{\{lines\} = } \begin{array}{ccccc}
\hline
& & & & \\
& & & & \\
& & & & \\
\end{array}
\]

Any one of these hexagons determines the triality uniquely if the order in which the edges are permuted is given. More precisely, given an orientation \(\partial\) of \(\langle Q \rangle\), an hexagon \((V, E)\) in \(\langle Q \rangle\) and an orientation of the circuit of edges of \(E\), there is a unique triality \((\tau, \partial)\) on \(\langle Q \rangle\) without absolute points that permutes the edges in \(E\) in the prescribed direction.
All geometric trialities

**Theorem** Let $\partial$ be a fixed orientation of $\langle Q \rangle$.

I There are 16 trialities $\tau, \partial$ with absolute points on $\langle Q \rangle$. All these trialities are conjugate under $\text{PGO}^+(\langle Q \rangle)$.

II There are 8 geometric trialities $\tau, \partial$ on $\langle Q \rangle$ without absolute points. These trialities are conjugate under the group $\text{PGO}^+(\langle Q \rangle)$.

**Consequence:**

- 2 isomorphism classes of geometric trialities;
- 2 isomorphism classes of 8-dimensional symmetric compositions;
- 2 conjugacy classes of trialitarian automorphisms;
Automorphisms

**Theorem** \((\tau, \partial)\) a geometric triality.

1) With absolute points.

\[
\text{Aut}(\tau, \partial) = D_{12} = S_2 \times S_3.
\]

2) Without absolute points.

\[
\text{Aut}(\tau, \partial) = \tilde{A}_4(\cong SL_2(\mathbb{F}_3)).
\]
Thank you for your attention!