partial hyperbolicity
and
topology of 3-manifolds

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celebrating Mike’s work
May 8, 2012
setting

**definition**

A partially hyperbolic diffeomorphism $f: M \to M$ is conservative (not always, but most of the time) on a closed Riemannian 3-manifold $M$. Anosov torus conjectures are related to this setting.
setting

- $M^3$ closed Riemannian 3-manifold
setting

- $M^3$ closed Riemannian 3-manifold
- $f : M \to M$ partially hyperbolic diffeomorphism
setting

- $M^3$ closed Riemannian 3-manifold
- $f : M \to M$ partially hyperbolic diffeomorphism
- $f$ conservative (not always, but most of the time)
partial hyperbolicity

\[ f : M^3 \to M^3 \text{ is partially hyperbolic} \]
partial hyperbolicity

\[ f : M^3 \to M^3 \text{ is partially hyperbolic} \]

\[ TM = E^s \oplus E^c \oplus E^u \]

- \( E^s \) \quad contracting
- \( E^c \) \quad intermediate
- \( E^u \) \quad expanding
Partial hyperbolicity

Let $f : M^3 \to M^3$ be a partial hyperbolic diffeomorphism. Then, the tangent bundle $TM$ can be decomposed as

$$TM = E^s \oplus E^c \oplus E^u$$

where $E^s$, $E^c$, and $E^u$ are 1-dimensional subbundles, respectively, representing the stable, center, and unstable directions.
example (conservative)

\[ f : \mathbb{T}^2 \times \mathbb{T}^1 \rightarrow \mathbb{T}^2 \times \mathbb{T}^1 \]
example (conservative)

\[ f : \mathbb{T}^2 \times \mathbb{T}^1 \rightarrow \mathbb{T}^2 \times \mathbb{T}^1 \]

such that

\[ f = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right) \times \text{id} \]
**example (conservative)**

\[ f : \mathbb{T}^2 \times \mathbb{T}^1 \rightarrow \mathbb{T}^2 \times \mathbb{T}^1 \]

such that

\[ f = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right) \times \text{id} \]
example (conservative)

\[ f : \mathbb{T}^2 \times \mathbb{T}^1 \rightarrow \mathbb{T}^2 \times \mathbb{T}^1 \]

such that

\[ f = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right) \times R_{\theta} \]
example (non-conservative)

\[ f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1 \]

such that

\[ f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times NPSP \]
open problems

problems

- ergodicity
open problems

problems

- ergodicity
- dynamical coherence
open problems

problems

- ergodicity
- dynamical coherence
- classification
most ph are ergodic

conjecture (pugh-shub)

partially hyperbolic diffeomorphisms

∪

$C^1$-open and $C^r$-dense set of ergodic diffeomorphisms
most ph are ergodic

**hertz-hertz-ures08**

partially hyperbolic diffeomorphisms

∪

$C^1$-open and $C^\infty$-dense set of ergodic diffeomorphisms
open problem

describe non-ergodic partially hyperbolic diffeomorphisms
open problem

describe non-ergodic partially hyperbolic diffeomorphisms

non-ergodicity
open problem

describe 3-manifolds supporting non-ergodic partially hyperbolic diffeomorphisms
An open problem is to describe 3-manifolds supporting non-ergodic partially hyperbolic diffeomorphisms.
An open problem is to describe 3-manifolds supporting non-ergodic partially hyperbolic diffeomorphisms.
conjecture

subliminal conjecture

most 3-manifolds
Anosov torus conjectures

non-ergodicity conjecture

Subliminal conjecture

Most 3-manifolds do not support non-ergodic partially hyperbolic diffeomorphisms
evidence

\(N\) 3-nilmanifold,
Then either $\mathcal{N} = \mathbb{T}^3$, or $\{\text{partially hyperbolic} \} \subseteq \{\text{ergodic} \}$ nilmanifolds are ergodic.
non-ergodicity

evidence

**hertz-hertz-ures08**

\[ N \text{ 3-nilmanifold, then either} \]

- \[ N = \mathbb{T}^3, \]
evidence

hertz-hertz-ures08

$N$ 3-nilmanifold, then either

- $N = \mathbb{T}^3$, or
- \{partially hyperbolic\} $\subset$ \{ergodic\}
evidence

hertz-hertz-ures08

$N$ 3-nilmanifold, then either

- $N = \mathbb{T}^3$, or
- $\{\text{partially hyperbolic} \} \subset \{\text{ergodic} \}$

nilmanifolds
Anosov torus

non-ergodicity

evidence

**hertz-hertz-ures08**

\( N \) 3-nilmanifold, then either

- \( N = \mathbb{T}^3 \), or
- \{partially hyperbolic\} \( \subset \) \{ergodic\}

**nilmanifolds**
non-ergodic conjecture

The only 3-manifolds supporting non-ergodic PH diffeomorphisms are:

1. the 3-torus,
2. the mapping torus of $-\text{id} : \mathbb{T}^2 \to \mathbb{T}^2$,
3. the mapping tori of hyperbolic automorphisms of $\mathbb{T}^2$. 
non-ergodic conjecture

The only 3-manifolds supporting non-ergodic PH diffeomorphisms are:

1. the 3-torus,
non-ergodic conjecture (hertz-hertz-ures)

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non-ergodic conjecture

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1. the 3-torus,
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
3. the mapping tori of hyperbolic automorphisms of $\mathbb{T}^2$
1 the 3-torus
non-ergodic conjecture

2 the mapping torus of $-id$
non-ergodic conjecture

3 the mapping torus of a hyperbolic automorphism
stronger non-ergodic conjecture

\[ f : M \to M \text{ non-ergodic partially hyperbolic diffeomorphism}, \text{ then} \]
stronger non-ergodic conjecture

If \( f : M \rightarrow M \) is a non-ergodic partially hyperbolic diffeomorphism, then

\[ \exists \text{ torus tangent to } E^s \oplus E^u \]
Integrability

\[ f: M^3 \to M^3 \text{ is partially hyperbolic} \]

\[ TM = E^s \oplus E^c \oplus E^u \]
integrability

$f : M^3 \to M^3$ is partially hyperbolic

\[ TM = E^s \oplus E^c \oplus E^u \]

\[ \mathcal{F}^s \]

\[ \mathcal{F}^u \]
integrability

\[ f : M^3 \to M^3 \text{ is partially hyperbolic} \]

\[
TM = E^s \oplus E^c \oplus E^u
\]

\[ \mathcal{F}^s \uparrow \quad \uparrow \quad \mathcal{F}^u \]

\[ ? \]
∃ invariant \( F \) tangent to \( E_s \oplus E_c \)

\[ \exists \text{invariant } F \text{ tangent to } E_c \oplus E_u \]

remark \( \Rightarrow \exists \text{invariant } F \text{ tangent to } E_c \)
∃ invariant $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
∃ invariant $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$

∃ invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$
1. \( \exists \) invariant \( \mathcal{F}^{cs} \) tangent to \( E^s \oplus E^c \)

2. \( \exists \) invariant \( \mathcal{F}^{cu} \) tangent to \( E^c \oplus E^u \)

remark

\( \Rightarrow \) \( \exists \) invariant \( \mathcal{F}^c \) tangent to \( E^c \)
open question

longstanding open question

\[ f : M^3 \to M^3 \text{ partially hyperbolic} \overset{?}{\Rightarrow} f \text{ dynamically coherent} \]
open question

longstanding open question

$f : M^3 \to M^3$ partially hyperbolic ? $f$ dynamically coherent

hertz-hertz-ures10

NO
counterexample

hertz-hertz-ures10

\[ \exists f : \mathbb{T}^3 \rightarrow \mathbb{T}^3 \text{ partially hyperbolic} \]
counterexample

\[ \exists f : \mathbb{T}^3 \to \mathbb{T}^3 \text{ partially hyperbolic} \]

- non-dynamically coherent
counterexample

exists f : T^3 → T^3 partially hyperbolic
  - non-dynamically coherent
  - non-conservative
∃ f : $\mathbb{T}^3 \to \mathbb{T}^3$ partially hyperbolic
- non-dynamically coherent
- non-conservative
- “robust”
open problem

describe 3-manifolds supporting non-dynamically coherent examples
**open problem**

*describe 3-manifolds supporting non-dynamically coherent examples*

**3-manifolds**

- **DC**
- **non DC**
open problem

describe 3-manifolds supporting non-dynamically coherent examples
**non-dynamically coherent conjecture**

<table>
<thead>
<tr>
<th>non-dynamically coherent conjecture (hertz-hertz-ures)</th>
</tr>
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<tbody>
<tr>
<td>$f : M^3 \rightarrow M^3$ non-dynamically coherent,</td>
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</tbody>
</table>
non-dynamically coherent conjecture

non-dynamically coherent conjecture (hertz-hertz-ures)

\( f : M^3 \rightarrow M^3 \) non-dynamically coherent, then \( M \) is either:
non-dynamically coherent conjecture

non-dynamically coherent conjecture (hertz-hertz-ures)

\[ f : M^3 \rightarrow M^3 \text{ non-dynamically coherent}, \]

then \( M \) is either:

1. \( T^3 \)

3-manifolds
non-dynamically coherent conjecture

non-dynamically coherent conjecture (hertz-hertz-ures)

\( f : M^3 \rightarrow M^3 \) non-dynamically coherent, then \( M \) is either:

1. \( \mathbb{T}^3 \)
2. the mapping torus of \(-id\)

3-manifolds
non-dynamically coherent conjecture

non-dynamically coherent conjecture (hertz-hertz-ures)

\[ f : M^3 \to M^3 \text{ non-dynamically coherent,} \]

then \( M \) is either:

1. \( \mathbb{T}^3 \)
2. the mapping torus of \(-id\)
3. the mapping torus of a hyperbolic automorphism

3-manifolds
stronger non-dynamically coherent conjecture

\[ f : M^3 \rightarrow M^3 \text{ non-dynamically coherent, then either} \]
stronger non-dynamically coherent conjecture

\( f : \mathcal{M}^3 \to \mathcal{M}^3 \) non-dynamically coherent, then either

- \( \exists \) torus tangent to \( E^c \oplus E^u \), or
stronger non-dynamically coherent conjecture

\[ f : M^3 \rightarrow M^3 \text{ non-dynamically coherent, then either} \]

- \( \exists \text{ torus tangent to } E^c \oplus E^u \), or
- \( \exists \text{ torus tangent to } E^s \oplus E^c \)
intermediate conjecture

\[ f \text{ volume preserving} \Rightarrow f \text{ dynamically coherent} \]
potrie11

\[ f : \mathbb{T}^3 \to \mathbb{T}^3 \] non-dynamically coherent, then
Anosov torus

Dynamical coherence

Evidence

potrie11

\[ f : \mathbb{T}^3 \rightarrow \mathbb{T}^3 \text{ non-dynamically coherent, then} \]

\[ \exists \text{ torus tangent to } E^c \oplus E^u, \text{ or} \]
potrie11

\[ f : \mathbb{T}^3 \to \mathbb{T}^3 \text{ non-dynamically coherent, then} \]

- \( \exists \) torus tangent to \( E^c \oplus E^u \), or
- \( \exists \) torus tangent to \( E^s \oplus E^c \)
examples of ph dynamics

known ph dynamics in dimension 3
examples of ph dynamics

**known ph dynamics in dimension 3**

- perturbations of time-one maps of Anosov flows
examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows
- certain skew-products
examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows
- certain skew-products
- certain DA-maps
examples of ph dynamics

known ph dynamics in dimension 3
- perturbations of time-one maps of Anosov flows
- certain skew-products
- certain DA-maps

new example
- non-dynamically coherent example
question

are there more examples?
classification conjecture (pujals01)

If \( f : M^3 \to M^3 \) is a transitive partially hyperbolic diffeomorphism, then \( f \) is (finitely covered by) either

1. a perturbation of a time-one map of an Anosov flow
2. a skew-product
3. a DA-map
If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then $f$ is (finitely covered by) either
1. a perturbation of a time-one map of an Anosov flow
If $f : M^3 \rightarrow M^3$ is a transitive partially hyperbolic diffeomorphism, then $f$ is (finitely covered by) either

1. a perturbation of a time-one map of an Anosov flow
2. a skew-product
If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then $f$ is (finitely covered by) either

1. a perturbation of a time-one map of an Anosov flow
2. a skew-product
3. a DA-map
If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then $f$ is leafwise conjugate to an Anosov flow leafwise conjugate to a skew-product with linear base leafwise conjugate to an Anosov map in $T^3$. 

classification conjecture (hhu)
**classification conjecture (hhu)**

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then $f$ is

- a perturbation of a time-one map of an Anosov flow,
If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then $f$ is

1. a perturbation of a time-one map of an Anosov flow,
2. a skew-product, or
If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then $f$ is

1. a perturbation of a time-one map of an Anosov flow,
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classification conjecture (hhu)

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then $f$ is

1. leafwise conjugate to an Anosov flow
2. a skew-product, or
3. a DA-map.
If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then $f$ is

1. leafwise conjugate to an Anosov flow
2. leafwise conjugate to a skew-product with linear base
3. a DA-map.
classification conjecture (hhu)

If $f : M^3 \rightarrow M^3$ is partially hyperbolic and dynamically coherent, then $f$ is

1. leafwise conjugate to an Anosov flow
2. leafwise conjugate to a skew-product with linear base
3. leafwise conjugate to an Anosov map in $\mathbb{T}^3$. 
Anosov torus

problems

- ergodicity
Anosov torus problems

- ergodicity
- dynamical coherence
problems

- ergodicity
- dynamical coherence
- classification
Anosov torus

problems

- ergodicity
- dynamical coherence
- classification

→

Anosov torus
Anosov torus

Anosov torus $T$ embedded 2-torus
Anosov torus

- $T$ embedded 2-torus
- $\exists f : M \to M$ s.t.
Anosov torus

- $T$ embedded 2-torus
- $\exists f : M \to M$ s.t.
  $$f(T) = T$$
Anosov torus

- $T$ embedded 2-torus
- $\exists f : M \to M$ s.t.
  - $f(T) = T$
  - $f|_T$ isotopic to Anosov
invariant tori in PH dynamics

\[ T \text{ invariant torus tangent to} \]
invariant tori in PH dynamics

$T$ invariant torus tangent to

$E^s \oplus E^u$
invariant tori in PH dynamics

\[ T \text{ invariant torus tangent to} \]

\[ E^s \oplus E^u \]

\[ E^c \oplus E^u \]
invariant tori in PH dynamics

$T$ invariant torus tangent to

- $E^s \oplus E^u$
- $E^c \oplus E^u$
- $E^s \oplus E^u$
invariant tori in PH dynamics

$T$ invariant torus tangent to

$E^s \oplus E^u$
$E^c \oplus E^u$  $\Rightarrow$  $T$ Anosov torus

$E^s \oplus E^u$
conjectures

Anosov torus conjectures

Then $M$ is either $T^3$ - the mapping torus of $-\text{id}: T^2 \to T^2$ or $T^3$ - the mapping torus of a hyperbolic map of $T^2$.

Stronger conjecture: $\exists$ Anosov torus tangent to $E_s \oplus E_u$ or $E_c \oplus E_u$. 
conjectures

non-ergodic conjecture

\( f : M \rightarrow M \) non-ergodic partially hyperbolic
conjectures

non-ergodic conjecture

\( f : M \to M \) non-ergodic partially hyperbolic

non-dyn. coh. conjecture

\( f : M \to M \) non-dyn. coherent partially hyperbolic

then \( M \) is either

1. the mapping torus of \(-\text{id} : T^2 \to T^2\)

or

3. the mapping torus of a hyperbolic map of \( T^2 \)

stronger conjecture

\( \exists \) Anosov torus tangent to \( E^s \oplus E^u \)

stronger conjecture

\( \exists \) Anosov torus tangent to \( E^c \oplus E^u \) or \( E^s \oplus E^c \)
conjectures

non-ergodic conjecture

\( f : M \to M \) non-ergodic partially hyperbolic

non-dyn. coh. conjecture

\( f : M \to M \) non-dyn. coherent partially hyperbolic

then \( M \) is either

1. \( \mathbb{T}^3 \)
conjectures

non-ergodic conjecture
\[ f : M \to M \text{ non-ergodic partially hyperbolic} \]

non-dyn. coh. conjecture
\[ f : M \to M \text{ non-dyn. coherent partially hyperbolic} \]

then \( M \) is either

1. \( \mathbb{T}^3 \)
2. the mapping torus of \(-id : \mathbb{T}^2 \to \mathbb{T}^2\)
conjectures

non-ergodic conjecture
\[ f : M \to M \text{ non-ergodic partially hyperbolic} \]

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\[ f : M \to M \text{ non-dyn. coherent partially hyperbolic} \]

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1. \( \mathbb{T}^3 \)
2. the mapping torus of \(-id : \mathbb{T}^2 \to \mathbb{T}^2\)
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conjectures

**non-ergodic conjecture**

\[ f : M \to M \text{ non-ergodic partially hyperbolic} \]

**non-dyn. coh. conjecture**

\[ f : M \to M \text{ non-dyn. coherent partially hyperbolic} \]

**then \( M \) is either**

1. \( \mathbb{T}^3 \)
2. the mapping torus of \(-id : \mathbb{T}^2 \to \mathbb{T}^2\)
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**stronger conjecture**

\[ \exists \text{ Anosov torus tangent to } E^s \oplus E^u \]
conjectures

non-ergodic conjecture
\( f : M \to M \) non-ergodic partially hyperbolic

then \( M \) is either
1. \( \mathbb{T}^3 \)
2. the mapping torus of \(-id : \mathbb{T}^2 \to \mathbb{T}^2\)
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non-dyn. coh. conjecture
\( f : M \to M \) non-dyn. coherent partially hyperbolic

stronger conjecture
\( \exists \) Anosov torus tangent to \( E^s \oplus E^u \)

stronger conjecture
\( \exists \) Anosov torus tangent to \( E^c \oplus E^u \) or \( E^s \oplus E^c \)
why stronger conjectures?

$hertz$-$hertz$-$ures11$

$M$ irreducible contains an Anosov torus,
why stronger conjectures?

hertz-hertz-ures11

\( M \) irreducible contains an Anosov torus, then \( M \) is either
why stronger conjectures?

hertz-hertz-ures11

\( M \) irreducible contains an Anosov torus, then \( M \) is either

1. \( T^3 \)

3-manifolds
why stronger conjectures?

**hertz-hertz-ures11**

*M* irreducible contains an Anosov torus, then *M* is either

1. $\mathbb{T}^3$
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$

**3-manifolds**
why stronger conjectures?

hertz-hertz-ures11

$M$ irreducible contains an Anosov torus, then $M$ is either

1. $T^3$
2. the mapping torus of $-id: T^2 \to T^2$
3. a mapping torus of a hyperbolic automorphism of $T^2$

3-manifolds
remark

\[ f : M^3 \rightarrow M^3 \text{ partially hyperbolic} \Rightarrow M \text{ irreducible} \]
remark

\[ f : M^3 \rightarrow M^3 \text{ partially hyperbolic } \Rightarrow M \text{ irreducible} \]

why
remark

$f : M^3 \to M^3$ partially hyperbolic $\Rightarrow M$ irreducible

why

- Rosenberg68
remark

\[
f : M^3 \to M^3 \text{ partially hyperbolic} \Rightarrow M \text{ irreducible}
\]

why

- Rosenberg68
- Burago-Ivanov08
reduced theorem

- $N^3$ irreducible manifold with boundary
reduced theorem

- $N^3$ irreducible manifold with boundary
- $\partial N$ consists of Anosov tori
reduced theorem

- $N^3$ irreducible manifold with boundary
- $\partial N$ consists of Anosov tori

⇒

$$N = \mathbb{T}^2 \times [0, 1]$$
brief sketch of the proof

JSJ-decomposition
brief sketch of the proof

**JSJ-decomposition**

- $N^3$ in our hypotheses
brief sketch of the proof

JSJ-decomposition

- $N^3$ in our hypotheses
- $\exists T_1, \ldots, T_n$ incompressible tori
brief sketch of the proof

**JSJ-decomposition**

- $N^3$ in our hypotheses
- $\exists \ T_1, \ldots, T_n$ incompressible tori
- such that each component of $M \setminus T_1 \cup \cdots \cup T_n$
  - atoroidal, or
brief sketch of the proof

**JSJ-decomposition**

- $N^3$ in our hypotheses
- $\exists \, T_1, \ldots, T_n$ incompressible tori
- such that each component of $M \setminus T_1 \cup \cdots \cup T_n$
  1. atoroidal, or
  2. Seifert manifold
brief sketch of the proof

JSJ-decomposition

- $N^3$ in our hypotheses
- $\exists \ T_1, \ldots, \ T_n$ incompressible tori
- such that each component of $M \setminus T_1 \cup \cdots \cup T_n$
  - atoroidal, or
  - Seifert manifold 🌟
brief sketch of the proof

remark
brief sketch of the proof

remark

- $N$ Seifert manifold
brief sketch of the proof

remark

- $N$ Seifert manifold
- $\partial N$ are Anosov tori
brief sketch of the proof

remark

- $N$ Seifert manifold
- $\partial N$ are Anosov tori
- $\Rightarrow \exists$ 2 Seifert fibrations non-isotopic on $\partial N$
brief sketch of the proof

classic lemma

- \( \exists \) 2 Seifert fibrations non-isotopic on \( \partial N \)
brief sketch of the proof

**classic lemma**

- $\exists$ 2 Seifert fibrations non-isotopic on $\partial N$
- $\Rightarrow$ $N$ is
brief sketch of the proof

classic lemma

- $\exists$ 2 Seifert fibrations non-isotopic on $\partial N$
- $\Rightarrow N$ is
  - $\mathbb{D}^2 \times S^1$ the solid torus
brief sketch of the proof

classic lemma

- \exists 2 \text{ Seifert fibrations non-isotopic on } \partial N
- \Rightarrow N \text{ is}
  1. \mathbb{D}^2 \times S^1 \text{ the solid torus}
  2. S^1 \times S^1 \times [0, 1] \text{ the twisted } I\text{-bundle over the Klein bundle}
brief sketch of the proof

**classic lemma**

- 2 Seifert fibrations non-isotopic on $\partial N$
- $\Rightarrow N$ is
  1. $\mathbb{D}^2 \times S^1$ the solid torus
  2. $S^1 \times S^1 \times [0, 1]$ the twisted $I$-bundle over the Klein bundle
  3. $T^2 \times [0, 1]$ the torus cross the interval
brief sketch of the proof

1. \( N = \) solid torus
brief sketch of the proof

1. $N = \text{solid torus}$

\[ \partial N = T \]
brief sketch of the proof

$$\textcircled{2} N_S = \text{twisted } I\text{-bundle over } K$$
brief sketch of the proof

2 $N_S = \text{twisted } I\text{-bundle over } K$

$\partial N = T$

©Ken Baker
brief sketch of the proof

3 \[ N = T \times [0, 1] \]
brief sketch of the proof

3 $N = T \times [0, 1]$

$\partial N = T \sqcup T$
brief sketch of the proof

final lemma

\[ \partial N \text{ consist of Anosov tori} \]
brief sketch of the proof

final lemma

- $\partial N$ consist of Anosov tori
- $\implies \partial N \neq T$
brief sketch of the proof

final lemma
- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$

why
- $i : H_1(\partial N) \hookrightarrow H_1(N)$
brief sketch of the proof

final lemma
- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$

why
- $i : H_1(\partial N) \hookrightarrow H_1(N)$
- $\text{rank}(\ker(i)) = \frac{1}{2} \text{rank}(H_1(\partial N))$
brief sketch of the proof

final lemma
- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$

why
- $i : H_1(\partial N) \hookrightarrow H_1(N)$
- $\text{rank}(\ker(i)) = \frac{1}{2} \text{rank}(H_1(\partial N))$

rank
- rank=“dimension"
final lemma

- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$
**final lemma**

- $\partial N$ consist of Anosov tori

- $\Rightarrow \partial N \neq T$

**possibilities**

$\partial N = T$
final lemma

- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$

possibilities

$\partial N = T$
**final lemma**

- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$

**possibilities**

- $\partial N = T$
- $\partial N = T$
- $\partial N = T \sqcup T$
final lemma

- $\partial N$ consist of Anosov tori
- $\Rightarrow \partial N \neq T$

possibilities

$$\partial N = T \sqcup T$$
**final lemma**

- $\partial N$ consist of Anosov tori

- $\Rightarrow \partial N \neq T$

**possibilities**

$\partial N = T \sqcup T$
thank you

THANK YOU MIKE!