“Metric Aspects in Algebraic Geometry, on the Average”.
(Celebrating the work of Mike Shub)

Luis M. Pardo

May, 2012

1Univ. de Cantabria.
Mike’s ideas have strongly influenced my work in the last decade.
Mike’s ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.
Mike’s ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale’s 17th Problem: [Beltrán-P., 2009]...
Mike’s ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale’s 17th Problem: [Beltrán-P., 2009]...

But these are already “old” mathematical results.
Mike’s ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale’s 17th Problem: [Beltrán-P., 2009]...

But these are already “old” mathematical results. I wanted something fresh, specifically oriented for this conference in Mike’s honor.
Mike’s ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale’s 17th Problem: [Beltrán-P., 2009]...

But these are already “old” mathematical results. I wanted something fresh, specifically oriented for this conference in Mike’s honor.

Thus, I tried to work on some (maybe modest and preliminary) results based on ideas from Mike’s work.
Some Self-Constraints
Some Self-Constraints

* No Condition Number.
Some Self-Constraints

* No Condition Number.
* No Complexity.
Some Self-Constraints

* No Condition Number.
* No Complexity.
* No Homotopy/Path Continuation Methods.
Some Self-Constraints

* No Condition Number.
* No Complexity.
* No Homotopy/Path Continuation Methods.
* No Polynomial System Solving.

Main outcome of this manuscript:

**Theorem**

\[
\frac{\text{vol}\{x \in B(0, r) : |Q(x)| < \varepsilon\}}{\text{vol}[B(0, r)]} \leq C_Q \frac{\varepsilon^{1/d}}{r},
\]

where \( Q \in \mathbb{R}[X_1, \ldots, X_n] \) is a polynomial of degree at most \( d \) and \( B(0, r) \) is the ball of radius \( r \) centered at the origin.
Plan of the Talk

- Expected Growth of Polynomials
- Expected Minimum Separation
- Expected Distance between two Complete Intersections
- On the Height of the Multi-variate Resultant Variety

Transforming this outcome into a recipe for this conference

Put something concerning the growth of the absolute value of multivariate polynomials.
Put something concerning the growth of the absolute value of multivariate polynomials.

Add some average and probability.
Transforming this outcome into a recipe for this conference

Put something concerning the growth of the absolute value of multivariate polynomials.

Add some average and probability.

And, finally, add some algebraic varieties and metrics and see what happens...
Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike’s double fibration technique).
Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike’s double fibration technique).
- On the Expected Distance between two complex projective varieties (same technique).
Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike’s double fibration technique).
- On the Expected Distance between two complex projective varieties (same technique).
- On the Expected average height of resultants:
Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike’s double fibration technique).
- On the Expected Distance between two complex projective varieties (same technique).
- On the Expected average height of resultants: An Arithmetic Poisson Formula for the Multi-variate Resultant.
Basic Notations (I)

*(d):= (d_{1}, \ldots, d_{m}) a list of degrees.
* \{X_{0}, \ldots, X_{n}\}:= A list of variables
* \mathcal{H}^{(m)}_{(d)} := Lists (f_{1}, \ldots, f_{m}) of \textit{m} complex homogeneous polynomials of respective degrees \text{deg}(f_{i}) = d_{i}.
* \mathcal{P}^{(m)}_{(d)} := Affine polynomials in \{X_{1}, \ldots, X_{n}\} with \text{deg}(f_{i}) \leq d_{i}.
* \mathcal{D}(d):= \prod_{i=1}^{m} d_{i} the Bézout number.
* N := the complex dimension of \mathcal{H}^{(m)}_{(d)} is N + 1.
Basic Notations (II)

*\(\mathbb{P}_n(\mathbb{C}) := \mathbb{P}(\mathbb{C}^{n+1})\), the projective complex space,

*\(d_R(x, y) := \) the Riemannian distance between two points \(x, y \in \mathbb{P}(\mathbb{C}^{n+1})\) and

\[
d_\mathbb{P}(x, y) := \sin d_R(x, y),
\]

the “projective” distance.

*\(V_\mathbb{P}(f) := \) for \(f \in \mathcal{H}_d^{(m)}\), the projective variety (in \(\mathbb{P}_n(\mathbb{C})\)) of the common zeros of polynomials in the list \(f\).

*\(V_\mathbb{A}(f) := \) for \(f \in \mathcal{P}_d^{(m)}\), the affine variety (in \(\mathbb{C}^n\)) of the common zeros of polynomials in the list \(f\).
An Hermitian form which is also an expectation:

$$\|f\|_\Delta^2 := \binom{d + n}{n} \frac{1}{\text{vol}[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^2 d\nu_S(z) = \binom{d + n}{n} E_{S^{2n+1}}[|f|^2].$$
An Hermitian form which is also an expectation:

\[ \|f\|_\Delta^2 := \binom{d+n}{n} \frac{1}{\text{vol}[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^2 d\nu_{S}(z) = \binom{d+n}{n} E_{S^{2n+1}}[\|f\|^2]. \]

* \( S(\mathcal{H}_d^{(m)}) \) := the sphere of radius one in \( \mathcal{H}_d^{(m)} \) with respect to Bombieri’s norm \( \| \cdot \|_\Delta \).
Expected Growth of Polynomials
An almost immediate statement

An Estimate

Assume $\text{deg}(f) = d \in 2\mathbb{Z}$ is of even degree. Then,

$$E_{\gamma}[|f(z)|] \leq ||f||_{\Delta} \left( \sum_{k=0}^{d/2} \binom{d/2}{k} \frac{2^{k-n-1}\Gamma(n+k+1)}{\pi^n\Gamma(n+1)} \right),$$

where $E_{\gamma}$ is the expectation with respect to the Gaussian distribution in $\mathbb{C}^n$. 
Let’s look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \rightarrow \mathbb{P}_n(\mathbb{C})$. 
Let’s look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$. 
Let’s look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \rightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$, and change the role of $\|f\|_{\Delta}$. 
Let’s look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull-back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \rightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$, and change the role of $||f||_\Delta$. Then, study:

$$\mathcal{E} := E_{f \in S(P_d^{(n)})}[E_{\mathbb{C}^n} [\log |f|]],$$

where $S(P_d^{(n)})$ is the sphere of radius one with respect to Bombieri–Weil norm.
Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \to \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$, and change the role of $||f||_\Delta$. Then, study:

$$\mathcal{C} := E_{f \in \mathcal{S}(P_d^{(n)})} [E_{\mathbb{C}^n} [\log |f|]] ,$$

where $\mathcal{S}(P_d^{(n)})$ is the sphere of radius one with respect to Bombieri–Weil norm.

This is the average value of something similar to the marking time in [Blum-Shub, 86].
Proposition

*With these notations we have:*

\[ E := \frac{1}{2} (dH_n - H_R), \]

*where \( H_r \) is the \( r \)-th harmonic number and \( R := \binom{d+n}{n} \).*
Proposition

*With these notations we have:*

\[ \mathcal{E} := \frac{1}{2} (dH_n - H_R), \]

where \( H_r \) is the \( r \)-th harmonic number and \( R := \binom{d+n}{n} \).

Recall that

\[ H_r \approx \log(r) + \gamma + O\left(\frac{1}{r}\right), \]

where \( \gamma \) is Euler-Mascheroni number and, hence

\[ \mathcal{E} \approx \frac{d}{2} \log \left( \frac{dn}{d+n} \right). \]
Corollary

\[ \mathcal{E} := E_{f \in \mathbb{S}(P_d^{(n)})}[E_{\mathbb{C}^n} |f^{-1}|] \geq e^{-\frac{d}{2}(H_n - H_R/d)}, \]
Other “potential” applications (not finished yet)

Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slightly modified...
Other “potential” applications (not finished yet)

Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slightly modified... Hints on $I(f)$, for $f \in \mathcal{H}^{(n)}_{(d)}$, and:

$$I(f) := \int_{S^{2n+1}} e^{\frac{||f(z)||^2}{2}} \frac{||f(z)||^{2n-1}}{||f(z)||^{2n-1}} dz.$$
Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slightly modified... Hints on $I(f)$, for $f \in \mathcal{H}^{(n)}_{(d)}$, and:

$$I(f) := \int_{S^{2n+1}} e^{\frac{||f(z)||^2}{2}} \frac{||f(z)||^{2n-1}}{||f(z)||^{2n-1}} dz.$$ 

A good “hint” could be the Expectation $E_{S(\mathcal{H}_{(d)})}[I(f)]$ and this is given by the following:
Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slightly modified... Hints on $I(f)$, for $f \in \mathcal{H}^{(n)}_{(d)}$, and:

$$I(f) := \int_{S^{2n+1}} \frac{e^{\frac{||f(z)||^2}{2}}}{||f(z)||^{2n-1}} dz.$$ 

A good “hint” could be the Expectation $E_{\mathcal{S}(\mathcal{H}_{(d)})}[I(f)]$ and this is given by the following:

$$E_{\mathcal{S}(\mathcal{H}_{d})}[I(f)] = \left( \sum_{k=0}^{\infty} \frac{\nu_{2n+1}}{k!2^k} \left( \sum_{j=0}^{\infty} \frac{(2k - n + 1)^j}{j!} E_{\mathcal{S}(\mathcal{H}_{(d)})}[E_{S^{2n+1}}[\log^j ||f||]] \right) \right).$$
On the average separation of the solutions
Double fibration (introduced and used by M. Shub and S. Smale in their Bézout series) with deep and interesting consequences, when combined with Federer’s co-area formula.
A Technique: solution varieties like as desing. à la Room-Kempf (I)

Double fibration (introduced and used by M. Shub and S. Smale in their Bézout series) with deep and interesting consequences, when combined with Federer’s co-area formula.

We consider the (smooth) solution variety:

\[ V^{(m)}_{(d)} = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}) : \zeta \in V_{\mathbb{P}}(f)\} \subseteq \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}). \]
A Technique: solution varieties like as desing. à la Room-Kempf (I)

Double fibration (introduced and used by M. Shub and S. Smale in their Bézout series) with deep and interesting consequences, when combined with Federer’s co-area formula.

We consider the (smooth) solution variety:

\[ V^{(m)}_{(d)} = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}) : \zeta \in V_{\mathbb{P}}(f)\} \subseteq \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}). \]

and we consider the two canonical projections:

\[ V^{(m)}_{(d)} \]

\[ \pi_1 \quad \pi_2 \]

\[ \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \quad \mathbb{P}_n(\mathbb{C}) \]
A Technique: solution variaties like as desing. à la Room-Kempf (II)

* \( \pi_2^{-1}(x) := \) is a “linear” (of co–dimension \( m \)) in \( \mathbb{P}(\mathcal{H}(m)) \).

* \( \pi_1^{-1}(f) := \) is the set of common zeros \( V_{\mathbb{P}}(f_1, \ldots, f_m) \) and it is “generically” a smooth projective variety of co–dimension \( m \).

* For \( m \leq n \), \( \pi_1 \) is onto.
Plan of the Talk

Expected Growth of Polynomials

Expected Minimum Separation

Expected Distance between two Complete Intersections

On the Height of the Multi-variate Resultant Variety

A Technique: solution varieties like as designing. à la Room-Kempf (II)

* $\pi_2^{-1}(x) :=$ is a “linear” (of co–dimension $m$) in $\mathbb{P}(\mathcal{H}_d^m)$.

* $\pi_1^{-1}(f) :=$ is the set of common zeros $V_{\mathbb{P}}(f_1,\ldots,f_m)$ and it is “generically” a smooth projective variety of co–dimension $m$.

* For $m \leq n$, $\pi_1$ is onto.

Shub–Smale’s idea

Averaging in $\mathbb{P}(\mathcal{H}_d^m)$ may be translated to averaging in $\mathbb{P}_n(\mathbb{C})$ through this double fibration.
For a zero–dimensional complete intersection variety $V_\mathbb{P}(f) \subseteq \mathbb{P}_n(\mathbb{C})$, the separation among its zeros:

$$sep(f) := \min\{d_\mathbb{P}(\zeta, \zeta') : \zeta, \zeta' \in V_\mathbb{P}(f), \zeta \neq \zeta'\}.$$ 

Lower bounds for these quantity are due to many authors:

- The Davenport–Mahler-Mignotte lower bound for the univariate case: $\Omega(2^{-d^2})$
- Other authors (Dedieu, Emiris, Mourrain, Tsigaridas, ...) have also treated the multivariate case:

$$\Omega(2^{-D_d})$$

---

\[\text{In [Castro–Haegele–Morais, p., 01] we also exhibited examples where this lower bound is achieved}\]
Separation of Solutions: An algorithmic question

\* \( f \in \mathcal{H}(d) \)

\* \( z_1, z_2 \in \mathbb{P}_n(\mathbb{C}) \) that satisfy \( \alpha - \)Theorem ([Shub–Smale]):

\[ \alpha(f, z_1) \leq \alpha_0, \quad \alpha(f, z_2) \leq \alpha_0. \]

Decide whether:

\[ \lim_{k \to \infty} N_f(z_1) = \lim_{k \to \infty} N_f(z_2)? \]
Separation of Solutions: An algorithm

* $t \in \mathbb{N}$

eval

$$z_1^{(t)} := N_f^t(z_1), \quad z_2^{(t)} := N_f^t(z_2).$$

if $d_P(z_1^{(t)}, z_2^{(t)}) > \frac{2}{2^{2t-1}}$, then OUTPUT: They approach DIFFERENT zeros of $f$

else, OUTPUT: They approach the same zero of $f$

fi
Separation of Solutions: An algorithm

* \( t \in \mathbb{N} \)

\[
\text{eval} \\
\begin{align*}
z_1^{(t)} &:= N_f^t(z_1), \quad z_2^{(t)} := N_f^t(z_2). \\
\end{align*}
\]

if \( d_P(z_1^{(t)}, z_2^{(t)}) > \frac{2}{2^{2t-1}} \), then \text{OUTPUT: } \text{They approach DIFFERENT zeros of } f

else, \text{OUTPUT: } \text{They approach the same zero of } f

fi

The algorithm works provided that:

\[
sep(f) > \frac{4}{2^{2t-1}}.
\]
For a zero–dimensional complete Intersection variety $V_{\mathbb{P}}(f) \subseteq \mathbb{P}^n(\mathbb{C})$, the “average” separation:

$$sep_{av}(f) := \frac{1}{\mathcal{D}(d)(\mathcal{D}(d) - 1)} \sum_{\zeta, \zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'} d_{\mathbb{P}}(\zeta, \zeta').$$

Theorem

Then, the following inequality holds:

$$E_{\mathbb{P}(\mathcal{H}(d))}[sep_{av}] \geq \frac{1}{2} \sqrt{\frac{1}{d^3(N + 1/2)n}}.$$
What about the minimum separation of solutions?

\[ sep_{\text{min}}(f) := \min_{\zeta, \zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'} d_{\mathbb{P}}(\zeta, \zeta'). \]
What about the minimum separation of solutions?

$$sep_{\text{min}}(f) := \min_{\zeta, \zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'} d_{\mathbb{P}}(\zeta, \zeta').$$

**Theorem**

*The following inequality holds:*

$$E_{\mathbb{S}(\mathcal{H}(d))} [sep_{\text{min}}(f)] \geq \frac{1}{4eD(d) d^{3/2}} (N + 1/2)^{-1/2}.$$
On the expected distance between two Complete Intersections
Expected Distance between two Complete Intersections

Let us consider two projective complete intersection varieties:

\[ V_\mathbb{P}(f) := \{ z \in \mathbb{P}_n(\mathbb{C}) : f_i(z) = 0, 1 \leq i \leq m \}. \]

\[ V_\mathbb{P}(g) := \{ z \in \mathbb{P}_n(\mathbb{C}) : g_j(z) = 0, 1 \leq j \leq s \}. \]
Expected Distance between two Complete Intersections

Let us consider two projective complete intersection varieties:

\[ V_{\mathbb{P}}(f) := \{ z \in \mathbb{P}_n(\mathbb{C}) : f_i(z) = 0, 1 \leq i \leq m \}. \]

\[ V_{\mathbb{P}}(g) := \{ z \in \mathbb{P}_n(\mathbb{C}) : g_j(z) = 0, 1 \leq j \leq s \}. \]

Average distance between \( V_{\mathbb{P}}(f) \) and \( V_{\mathbb{P}}(g) \) as

\[ D_{av}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g)) := \frac{1}{vol[V_{\mathbb{P}}(f)]vol[V_{\mathbb{P}}(g)]} \int_{V_{\mathbb{P}}(f) \times V_{\mathbb{P}}(g)} d_{\mathbb{P}}(x, y) dV_{\mathbb{P}}(f) dV_{\mathbb{P}}(g). \]
Let us consider two projective complete intersection varieties:

\[ V_{\mathbb{P}}(f) := \{ z \in \mathbb{P}_n(\mathbb{C}) : f_i(z) = 0, 1 \leq i \leq m \}. \]

\[ V_{\mathbb{P}}(g) := \{ z \in \mathbb{P}_n(\mathbb{C}) : g_j(z) = 0, 1 \leq j \leq s \}. \]

Average distance between \( V_{\mathbb{P}}(f) \) and \( V_{\mathbb{P}}(g) \) as

\[ D_{av}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g)) := \frac{1}{vol[V_{\mathbb{P}}(f)]vol[V_{\mathbb{P}}(g)]} \int_{V_{\mathbb{P}}(f) \times V_{\mathbb{P}}(g)} d_{\mathbb{P}}(x, y)dV_{\mathbb{P}}(f)dV_{\mathbb{P}}(g). \]

**Theorem**

*With these notations, we have:*

\[ E_{f,g}[D_{av}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g))] = (1 - \frac{1}{n + 2}). \]
Expected Distance between two Complete Intersections (II)

Same notations. Assume $V_P(f)$ is zero–dimensional and $s \geq 1$ (i.e. $V_P(f) \cap V_P(g) = \emptyset$ a.e.).

Distance between $V_P(f)$ and $V_P(g)$ as

$$d_P(V_P(f), V_P(g)) := \min\{d_P(x, y) : x \in V_P(f), y \in V_P(g)\}.$$
Expected Distance between two Complete Intersections (II)

Same notations. Assume $V_P(f)$ is zero–dimensional and $s \geq 1$ (i.e. $V_P(f) \cap V_P(g) = \emptyset$ a.e.).

Distance between $V_P(f)$ and $V_P(g)$ as

$$d_P(V_P(f), V_P(g)) := \min\{d_P(x, y) : x \in V_P(f), y \in V_P(g)\}.$$  

**Theorem**

Assume $V_P(g)$ is of co–dimension $s$ and $\deg(V_P(g)) := D' = \prod_{i=1}^{s} d'_i$, we have

$$E_{f,g}[d_P(V_P(f), V_P(g))] \geq \frac{2s - 1}{D(d) \left(1 + 2 \left(\prod_{i=1}^{s} \frac{d'_i e^2}{s^2}\right)\right)},$$

Moreover, for $s \geq 3$, this may be rewritten as

$$E_{f,g}[d_P(V_P(f), V_P(g))] \geq \frac{2s - 1}{D(d) + 2\deg(V_P(g))D(d)}.$$
On the Height of the Multi-variate Resultant: an Arithmetic Poisson formula
Multi-variate Resultant (I)

We recall the “solution variety” in the case of over-determined systems:

\[ V_{(d)}^{(n+1)} = \{ (f, \zeta) \in \mathbb{P}(\mathcal{H}_{(d)}^{(n+1)}) \times \mathbb{P}_n(\mathbb{C}) : \zeta \in V_{\mathbb{P}(f)} \} \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{(n+1)}) \times \mathbb{P}_n(\mathbb{C}). \]

and we also consider the two canonical projections:

\[ \begin{array}{c}
\pi_1 \leftarrow \mathbb{P}(\mathcal{H}_{(d)}^{(n+1)}) \\
\pi_2 \rightarrow \mathbb{P}_n(\mathbb{C})
\end{array} \]
* $\pi_2^{-1}(x) :=$ is a “linear” (of co–dimension $n + 1$) in $\mathbb{P}(H^{(n+1)}_{(d)})$.

* $\pi_1^{-1}(f) :=$ is the set of common zeros $V_{\mathbb{P}}(f_0, \ldots, f_n)$ and it is either $\emptyset$ or “generically” a single point.

* $\pi_1(V^{(n+1)}_{(d)}) := R^{(n+1)}_{(d)}$ is an irreducible complex hyper–surface, usually known as the multi–variate resultant variety.

* Multi–variate resultant $Res^{(n+1)}_{(d)} :=$ is the multi–homogeneous irreducible polynomial which defines $R^{(n+1)}_{(d)}$. 

Multi-variate Resultant and Resultant Variety $R^{(n+1)}_d$ is a classical object in Elimination Theory (also Computational Algebraic Geometry, MEGA...).

It has been studied by many authors since XIX-th century with different approaches and variations: Bézout, Sylvester, Macaulay, Chow,..., and, more recently, Jouanolou, Chardin, Gelfand, Kapranov, Zelevinsky, Sturmfels, Rojas, Heintz, Giusti, Dickenstein, D’Andrea, Krick, Szanto, Sombra...
Multi-variate Resultant and Resultant Variety $R_{(d)}^{(n+1)}$ is a classical object in Elimination Theory (also Computational Algebraic Geometry, MEGA...).

It has been studied by many authors since XIX-th century with different approaches and variations: Bézout, Sylvester, Macaulay, Chow,..., and, more recently, Jouanolou, Chardin, Gelfand, Kapranov, Zelevinsky, Sturmfels, Rojas, Heintz, Giusti, Dickenstein, D’Andrea, Krick, Szanto, Sombra...

The list is too long to be complete...
Height of the Multi-Variate Resultant (I)

* Height of the multivariate resultant is an attempt to measure the length of the integer coefficients...
* Height of the multivariate resultant is an attempt to measure the length of the integer coefficients...

It has interesting applications both in complexity and arithmetic geometry.

With several variations (Chow forms, elimination polynomials, Arithmetic Nullstellensatz...) it has been studied by many authors: Nesterenko, Philippon, Krick, Sombra, D’Andrea, Rémond, P. ...
* Height of the multivariate resultant is an attempt to measure the length of the integer coefficients...

It has interesting applications both in complexity and arithmetic geometry.

With several variations (Chow forms, elimination polynomials, Arithmetic Nullstellensatz...) it has been studied by many authors: Nesterenko, Philippon, Krick, Sombra, D’Andrea, Rémond, P. ...

Multi–variate resultants satisfy a Poisson formula which is a helpful statement for the knowledge of its properties.

Here, we are modest: we focus on the arithmetic version of Poisson Formula, whose geometric (degree) property can be stated as follows:
“Geometric” Poisson Formula

Let \((d) \equiv (d_0, d_1, \ldots, d_n)\) be a degree list, and \((d') \equiv (d_1, \ldots, d_n)\).
“Geometric” Poisson Formula

Let \((d) := (d_0, d_1, \ldots, d_n)\) be a degree list, and \((d') := (d_1, \ldots, d_n)\).

Let \(Res^{(n+1)}_{(d)}\) be the resultant associated to \((d)\) in \(n + 1\) variables and \(Res^{(n)}_{(d')}\) the corresponding one associated to \((d')\).
"Geometric" Poisson Formula

Let \((d) := (d_0, d_1, \ldots, d_n)\) be a degree list, and \((d') := (d_1, \ldots, d_n)\).

Let \(\text{Res}^{(n+1)}(d)\) be the resultant associated to \((d)\) in \(n + 1\) variables and \(\text{Res}^{(n)}(d')\) the corresponding one associated to \((d')\).

For instance, Poisson’s Formula implies:

\[
\deg(\text{Res}^{(n+1)}(d)) = d_0 \deg(\text{Res}^{(n)}(d')) + D(d'),
\]

where \(D(d') := \prod_{i=1}^{n} d_i\) is the Bézout number.
Let \((d) := (d_0, d_1, \ldots, d_n)\) be a degree list, and \((d') := (d_1, \ldots, d_n)\).

Let \(Res_{(d)}^{(n+1)}\) be the resultant associated to \((d)\) in \(n + 1\) variables and \(Res_{(d')}^{(n)}\) the corresponding one associated to \((d')\).

For instance, Poisson’s Formula implies:

\[
\text{deg}(Res_{(d)}^{(n+1)}) = d_0 \cdot \text{deg}(Res_{(d')}^{(n)}) + D_{(d')},
\]

where \(D_{(d')} := \prod_{i=1}^{n} d_i\) is the Bézout number.

Inductively, we conclude:

\[
\text{deg}(Res_{(d)}^{(n+1)}) = \sum_{i=0}^{n} \prod_{j \neq i} d_j.
\]
We may define the logarithmic height of the multi–resultant variety either following any of the usual definitions [Philippon, 91], [Bost, Gillet, Soulé, 94], [Rémon, 01], [McKinnon, 01], [D’Andrea, Krick, Sombra, 11].... We just modify them by using the unitarily invariant height
\[ ht_u(R_{(d)}^{n+1}) : \]
We may define the logarithmic height of the multi–resultant variety either following any of the usual definitions [Philippon, 91], [Bost, Gillet, Soulé, 94], [Rémon,01], [McKinnon, 01], [D’Andrea, Krick, Sombra, 11]....
We just modify them by using the unitarily invariant height
\( h_{tu}(R_{(d)}^{(n+1)}) \):

The only difference with “usual” notions is that we take into account Bombieri’s metric in the logarithmic Mahler’s measure (instead of the usual Hermitian product).
Define the (unitarily invariant) logarithmic Mahler measure:

$$m_{\mathcal{G}^{(n+1)}(d)} (\text{Res}^{(n+1)}_{(d)}) := E_{\mathcal{G}^{(n+1)}(d)} \left[ \log |\text{Res}^{(n+1)}_{(d)}(f_0, \ldots, f_n)| \right]$$

where

$$\mathcal{G}^{(n+1)}(d) := \prod_{i=0}^{n} \mathcal{S}(H_{d_i}).$$
(Unitarily invariant) Logarithmic Mahler measure

Define the (unitarily invariant) logarithmic Mahler measure:

\[
m_{\mathcal{G}^{(n+1)}_{(d)}}(\text{Res}^{(n+1)}_{(d)}) := E_{\mathcal{G}^{(n+1)}_{(d)}} \left[ \log |\text{Res}^{(n+1)}_{(d)}(f_0, \ldots, f_n)| \right]
\]

where

\[
\mathcal{G}^{(n+1)}_{(d)} := \prod_{i=0}^{n} \mathcal{S}(H_{d_i}).
\]

For technical reasons, define:

\[
\mathcal{R}^{(n+1)}_{(d)} := \frac{ht_u(R^{(n+1)}_{(d)})}{D_{(d)}},
\]

Note that this quantity only depends of \((d) := (d_0, \ldots, d_n)\) the “degree” list.
With these notations we have:

\[ R^{(n+1)}_{(d)} = \prod_{i=1}^{n} \left( \frac{n}{d_i + n} \right) \left( R^{(n)}_{(d')} + I_{(d)} \right) + \frac{C}{d_0}, \]
Theorem

*With these notations we have:*

\[
\mathcal{R}_{(d)}^{(n+1)} = \prod_{i=1}^{n} \left( \frac{n}{d_i + n} \right) \left( \mathcal{R}_{(d')}^{(n)} + \mathcal{I}(d) \right) + \frac{\mathcal{E}}{d_0},
\]

*where* \((d') := (d_1, \ldots, d_n)\), *and*

\[
\frac{\mathcal{E}}{d_0} = \frac{1}{2} \left( H_n - \frac{H_R}{d_0} \right)
\]
With these notations we have:

$$R_{(d)}^{(n+1)} = \prod_{i=1}^{n} \left( \frac{n}{d_i + n} \right) \left( R_{(d')}^{(n)} + I_{(d)} \right) + \frac{E}{d_0},$$

where $(d') := (d_1, \ldots, d_n)$, and

$$\frac{E}{d_0} = \frac{1}{2} \left( H_n - \frac{H_R}{d_0} \right)$$

$$-\frac{1}{4} \leq I_{(d)} \leq 0.$$
Some corollaries (I)

Corollary

*With the same notations, we have:*

\[
|\mathcal{R}_{(d)}^{(n+1)} - \prod_{i=1}^{n} \left( \frac{n}{d_i + n} \right) \mathcal{R}_{(d')}^{(n)}| \leq \frac{1}{2} \log \left( \frac{d_0 n}{d_0 + n} \right) + O\left( \frac{1}{n} \right),
\]
The straightforward inductive argument yields

$$\mathcal{R}^{(n+1)}(d) \leq \frac{1}{2} \sum_{i=0}^{n} \log \left( \frac{d_i n}{d_i + n} \right) + O(1) \approx \frac{1}{2} n \log(n) + O(1),$$

and
Some Corollaries (II)

The straightforward inductive argument yields

\[ \mathcal{R}^{(n+1)}_{(d)} \leq \frac{1}{2} \sum_{i=0}^{n} \log \left( \frac{d_i n}{d_i + n} \right) + O(1) \approx \frac{1}{2} n \log(n) + O(1), \]

and

\[ ht_u \left( R^{(n+1)}_{(d)} \right) \leq \frac{D_{(d)}}{2} \left( \sum_{i=0}^{n} \log \left( \frac{d_i n}{d_i + n} \right) + c \right) \approx \frac{D_{(d)}}{2} (n \log(n) + c). \]
Controlling the growth (in the sense of [Blum-Shub, 86])

Let us consider now the Gaussian distribution $\gamma_\Delta$ in $\mathcal{H}^{n+1}_d$ induced by the Bombieri’s norm.

**Corollary**

*With these notations, we have:*

$$\text{Prob}_{\gamma_\Delta} \left[ \log | Res^{(n+1)}_d(f_0, \ldots, f_n) | \geq \varepsilon^{-1} + \sum_{i=0}^{n} \left( \prod_{j \neq i} d_j \right) \log ||f||_\Delta \right] \leq \frac{\mathcal{D}(d)}{2} (n \log(n) + c) \varepsilon,$$

*for some constant $c > 0$.*
Some corollaries (III)

* van der Waerden’s U-resultant $\chi_U$ is a classical object in Elimination Theory (some times called Chow form, Elimination Polynomial,...).
* Upper bounds for the complexity of computing $U$–resultants were shown in [Jerónimo-Krick-Sabia-Sombra, 03]
Some corollaries (III)

* van der Waerden’s U-resultant $\chi_U$ is a classical object in Elimination Theory (some times called Chow form, Elimination Polynomial,...).
* Upper bounds for the complexity of computing $U$–resultants were shown in [Jerónimo-Krick-Sabia-Sombra, 03]
* In [Heintz-Morgenstern, 93]: Computation of the $U$–resultant is \textbf{NP}–hard.
* van der Waerden’s U-resultant $\chi_U$ is a classical object in Elimination Theory (some times called Chow form, Elimination Polynomial,...).
* Upper bounds for the complexity of computing $U$–resultants were shown in [Jerónimo-Krick-Sabia-Sombra, 03]
* In [Heintz-Morgenstern, 93]: Computation of the $U$–resultant is \textbf{NP}–hard.

Modestly, we may immediate obtain:
Let $\mathcal{E}_U$ be the expected logarithmic Mahler’s measure of the $U$–resultant with respect to some projective variety determined by the degree list $(d') := (d_1, \ldots, d_n)$:

**Corollary**

*With these notations, we have*

$$E_{(d')}^{(n)} [m(\chi_U)] \leq \left( \prod_{i=1}^{n} d_i \right) (n \log(n) + c),$$
Finally, you may use the ideas by Hardy, Mordell, Davenport and others\textsuperscript{3} on the equidistribution of polynomial systems with Gaussian rational coefficients of bounded height in the projective space $\mathbb{P}(H_d)$ to conclude to conclude that

\textsuperscript{3}See also [Castro-Montaña-P.-San Martin, 02] or [P.-San Martin, 04].
Finally, you may use the ideas by Hardy, Mordell, Davenport and others\textsuperscript{3} on the equidistribution of polynomial systems with Gaussian rational coefficients of bounded height in the projective space $\mathbb{P}(H_d)$ to conclude that

**Corollary**

*The same bound holds for the expected Mahler’s measure of the $U$–resultant for random systems with Gaussian rational coefficients of bounded height and uniform distribution.*

\textsuperscript{3}See also [Castro-Montaña-P.-San Martin, 02] or [P.-San Martin, 04].
Forthcoming Tasks

These bounds are not satisfactory yet.
Forthcoming Tasks

* These bounds are not satisfactory yet. There must be improvements and more precise estimates...ongoing research...
These bounds are not satisfactory yet. There must be improvements and more precise estimates...ongoing research...

* Continue with studies on average properties of MAAG.
* Compare unitarily invariant height to other notions of height......
These bounds are not satisfactory yet. There must be improvements and more precise estimates...ongoing research...

* Continue with studies on average properties of MAAG.
* Compare unitarily invariant height to other notions of height......
* Continue exploration of the over–determined case...the main problem.
Forthcoming Tasks

Happy May’68, Mike!
Forthcoming Tasks

Happy May’68, Mike!

and thanks to all of you for your patience!