Volumes of polyhedra in hyperbolic and spherical spaces

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The calculation of the volume of a polyhedron in 3-dimensional space E^3, H^3, or S^3 is a very old and difficult problem. The first known result in this direction belongs to **Tartaglia (1499-1557)** who found a formula for the volume of Euclidean tetrahedron. Now this formula is known as Cayley-Menger determinant. More precisely, let be an Euclidean tetrahedron with edge lengths d_{ij}, $1 \leq i < j \leq 4$. Then $V = \mathrm{Vol}(T)$ is given by

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Note that V is a root of quadratic equation whose coefficients are integer polynomials in d_{ij}, $1 \leq i < j \leq 4$.

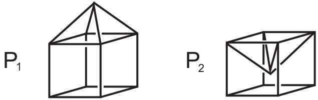
Introduction

Surprisely, but the result can be generalized on any Euclidean polyhedron in the following way.

Theorem 1 (I. Kh. Sabitov, 1996)

Let P be an Euclidean polyhedron. Then $V = Vol(P)$ is a root of an even degree algebraic equation whose coefficients are integer polynomials in edge lengths of P depending on combinatorial type of P only.

Example



(All edge lengths are taken to be 1)

Polyhedra P_1 and P_2 are of the same combinatorial type. Hence, $V_1 = Vol(P_1)$ and $V_2 = Vol(P_2)$ are roots of the same algebraic equation

$$a_0 V^{2n} + a_1 V^{2n-2} + \ldots + a_n V^0 = 0.$$

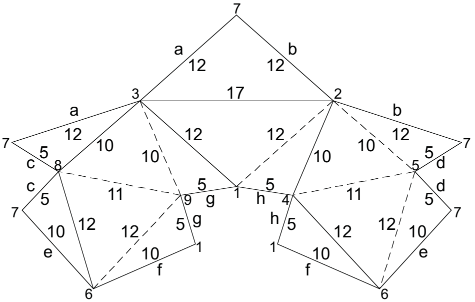
Introduction

Cauchy theorem (1813) states that if the faces of a convex polyhedron are made of metal plates and the polyhedron edges are replaced by hinges, the polyhedron would be rigid. In spite of this there are non-convex polyhedra which are flexible.

Bricard, 1897 (self-interesting flexible octahedron)

Connelly, 1978 (the first example of true flexible polyhedron)

The smallest example is given by Steffen (14 triangular faces and 9 edges).



Introduction

- Bellows Conjecture

Very important consequence of Sabitov's theorem is a positive solution of the Bellows Conjecture proposed by Dennis Sullivan.

Theorem 2 (R. Connelly, I. Sabitov and A. Walz, 1997)

All flexible polyhedra keep a constant volume as they are flexed.

It was shown by Victor Alexandrov (Novosibirsk, 1997) that Bellows Conjecture fails in the spherical space S^3. In the hyperbolic space H^3 the problem is still open.

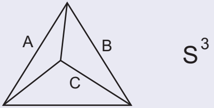
Recently, A.A. Gaifullin (2011) proved a four dimensional version of the Sabitov's theorem.

Any analog of Sabitov's theorem is unknown in both spaces S^3 and H^3.

Spherical orthoscheme

Theorem 3 (L. Schläfli)

The volume of a spherical orthoscheme with essensial dihedral angles A, B and C



is given by the formula $V = \frac{1}{4} S(A, B, C)$, *where*

$$S(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z) = \widehat{S}(x, y, z) =$$

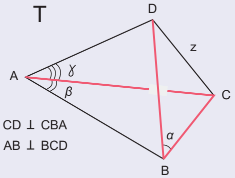
$$\sum_{m=1}^{\infty} \left(\frac{D - \sin x \sin z}{D + \sin x \sin z} \right)^m \frac{\cos 2mx - \cos 2my + \cos 2mz - 1}{m^2} - x^2 + y^2 - z^2$$

and $D \equiv \sqrt{\cos^2 x \cos^2 z - \cos^2 y}$.

The volume of a biorthogonal tetrahedron (orthoscheme) was calculated by Lobachevsky and Bolyai in H^3 and by Schläfli in S^3.

Theorem 4 (J. Bolyai)

The volume of hyperbolic orthoscheme T is given by the formula



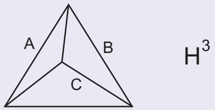
$$Vol\,(T) = \frac{\tan\gamma}{2\tan\beta}\int\limits_0^z \frac{z\sinh z\,dz}{\left(\frac{\cosh^2 z}{\cos^2\alpha}-1\right)\sqrt{\frac{\cosh^2 z}{\cos^2\gamma}-1}}.$$

Hyperbolic orthoscheme

The following theorem is the Coxeter's version of the Lobachevsky result.

Theorem 5 (Lobachevsky, Coxeter)

The volume of a hyperbolic orthoscheme with essential dihedral angles A, B and C



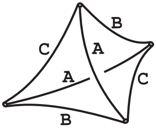
is given by the formula

$$V = \frac{i}{4} S(A, B, C),$$

where $S(A, B, C)$ is the Schläfli function.

Ideal polyhedra

Consider an ideal hyperbolic tetrahedron T with all vertices on the infinity



Opposite dihedral angles of ideal tetrahedron are equal to each other and $A + B + C = \pi$.

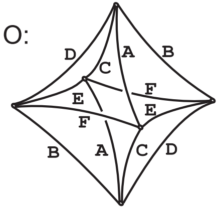
Theorem 6 (J. Milnor, 1982)

$Vol(T) = \Lambda(A) + \Lambda(B) + \Lambda(C)$, where $\Lambda(x) = -\int\limits_0^x \log|2\sin t|dt$ is the Lobachevsky function.

More complicated case with only one vertex on the infinity was investigated by **E. B. Vinberg (1993)**.

Ideal polyhedra

Let O be an ideal symmetric octahedron with all vertices on the infinity.



Then $C = \pi - A,\ D = \pi - B,\ F = \pi - E$ and the volume of O is given by

Theorem 7 (Yana Mohanty, 2002)

$$Vol\,(O) = 2\Big(\Lambda\left(\frac{\pi + A + B + E}{2}\right) + \Lambda\left(\frac{\pi - A - B + E}{2}\right)$$
$$+ \Lambda\left(\frac{\pi + A - B - E}{2}\right) + \Lambda\left(\frac{\pi - A + B - E}{2}\right)\Big).$$

Volume of tetrahedron

Despite of the above mentioned partial results, a formula the volume of an arbitrary hyperbolic tetrahedron has been unknown until very recently. The general algorithm for obtaining such a formula was indicated by **W.–Y. Hsiang (1988)** and the complete solution of the problem was given by **Yu. Cho and H. Kim (1999)**, **J. Murakami, M. Yano (2001)** and **A. Ushijima (2002)**.

In these papers the volume of tetrahedron is expressed as an analytic formula involving 16 Dilogarithm of Lobachevsky functions whose arguments depend on the dihedral angles of the tetrahedron and on some additional parameter which is a root of some complicated quadratic equation with complex coefficients.
A geometrical meaning of the obtained formula was recognized by **G. Leibon** from the point of view of the *Regge symmetry*. An excellent exposition of these ideas and a complete geometric proof of the volume formula was given by **Y. Mohanty (2003)**.

Volume of tetrahedron

We suggest the following version of the integral formula for the volume. Let $T = T(A, B, C, D, E, F)$ be a hyperbolic tetrahedron with dihedral angles A, B, C, D, E, F. We set

$V_1 = A + B + C,\ V_2 = A + E + F,\ V_3 = B + D + F,\ V_4 = C + D + E$ (for vertices)

$H_1 = A + B + D + E,\ H_2 = A + C + D + F,\ H_3 = B + C + E + F,\ H_4 = 0$ (for Hamiltonian cycles).

Theorem 8 (D. Derevnin and M., 2005)

The volume of a hyperbolic tetrahedron is given by the formula

$$Vol(T) = -\frac{1}{4}\int\limits_{z_1}^{z_2} \log \prod_{i=1}^{4} \frac{\cos\frac{V_i + z}{2}}{\sin\frac{H_i + z}{2}}\, dz,$$

where z_1 and z_2 are appropriate roots of the integrand.

Volume of tetrahedron

More precisely, the roots in the previous theorem are given by the formulas

$$z_1 = \arctan\frac{K_2}{K_1} - \arctan\frac{K_4}{K_3}, \;\; z_2 = \arctan\frac{K_2}{K_1} + \arctan\frac{K_4}{K_3}$$

and

$$K_1 = -\sum_{i=1}^{4}(\cos(S - H_i) + \cos(S - V_i)),$$

$$K_2 = \sum_{i=1}^{4}(\sin(S - H_i) + \sin(S - V_i)),$$

$$K_3 = 2(\sin A \sin D + \sin B \sin E + \sin C \sin F),$$

$$K_4 = \sqrt{K_1^2 + K_2^2 - K_3^2}, \qquad S = A + B + C + D + E + F.$$

Volume of tetrahedron

Recall that the Dilogarithm function is defined by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt.$$

We set $\ell(z) = \mathrm{Li}_2(e^{iz})$ and note that $\Im(\ell(z)) = 2\,\Lambda(\frac{z}{2})$.

The following result is a consequence of the above theorem.

Theorem 9 (J. Murakami, M. Yano, 2001)

$$Vol(T) = \tfrac{1}{2}\Im(U(z_1, T) - U(z_2, T)), \ \ where$$

$$\begin{aligned} U(z,T) = \frac{1}{2}(&\ell(z) + \ell(A+B+D+E+z) \\ &+\ell(A+C+D+F+z) + \ell(B+C+E+F+z) \\ &-\ell(\pi+A+B+C+z) - \ell(\pi+A+E+F+z) \\ &-\ell(\pi+B+D+F+z) - \ell(\pi+C+D+E+z)). \end{aligned}$$

More deep history

It is surprising that, more than a century ago, in 1906, the Italian mathematician G. Sforza found the formula for the volume of a non-Euclidean tetrahedron. This fact became known during a discussion of the author with J. M. Montesinos at the conference in El Burgo d Osma (Spain) in August 2006.

Let G be Gram matrix for hyperbolic tetrahedron T. We set $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is ij-th minor of matrix G.

Theorem 10 (G. Sforza, 1906)

The volume of a hyperbolic tetrahedron T is given by the following formula

$$Vol(T) = \frac{1}{4} \int\limits_{A_0}^{A} \log \frac{c_{34} - \sqrt{-\det G} \sin A}{c_{34} + \sqrt{-\det G} \sin A} dA,$$

where A_0 is a root of the equation $\det G = 0$.

- Proof of Sforza formula

We start with the the following theorem.

Theorem 11 (Jacobi)

Let $G = (a_{ij})_{i,j=1,\ldots,n}$ be an $n \times n$ matrix with $det\, G = \Delta$. Denote by $C = (c_{ij})_{i,j=1,\ldots,n}$ the matrix formed by elements $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is ij-th minor of matrix G. Then

$$det\,(c_{ij})_{i,j=1,\ldots,k} = \Delta^{k-1} det\,(a_{ij})_{i,j=k+1,\ldots,n}.$$

Sforza formula

Apply the theorem to Gram matrix G for $n = 4$ and $k = 2$

$$G = \begin{pmatrix} 1 & -\cos A & x & x \\ -\cos A & 1 & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}, C = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & c_{33} & c_{34} \\ x & x & c_{43} & c_{44} \end{pmatrix}.$$

We have $\quad c_{33}c_{44} - c_{34}^2 = \Delta(1 - \cos^2 A).$

By Cosine Rule

$$\cosh \ell_A = \frac{c_{34}}{\sqrt{c_{33}c_{44}}}, \text{ hence}$$

$$\sinh \ell_A = \sqrt{\frac{c_{34}^2 - c_{33}c_{44}}{c_{33}c_{44}}} = \frac{\sin A}{\sqrt{c_{33}c_{44}}}\sqrt{-\Delta}\,.$$

Sforza formula

Since $\exp(\pm\ell_A) = \cosh\ell_A \pm \sinh\ell_A$ we have

$$\exp(\ell_A) = \frac{c_{34} + \sin A\sqrt{-\Delta}}{\sqrt{c_{33}c_{44}}}, \quad \exp(-\ell_A) = \frac{c_{34} + \sin A\sqrt{-\Delta}}{\sqrt{c_{33}c_{44}}}.$$

Hence,

$$\exp(2\ell_A) = \frac{c_{34} + \sin A\sqrt{-\Delta}}{c_{34} - \sin A\sqrt{-\Delta}}, \quad \text{and} \quad \ell_A = \frac{1}{2}\log\frac{c_{34} + \sin A\sqrt{-\Delta}}{c_{34} - \sin A\sqrt{-\Delta}}.$$

By the Schläfli formula

$$-dV = \frac{1}{2}\sum_{\alpha}\ell_\alpha d\alpha, \quad \alpha \in \{A, B, C, D, E, F\}$$

$$V = \int\limits_{A_0}^{A}(-\frac{\ell_A}{2})dA = \frac{1}{4}\int\limits_{A_0}^{A}\log\frac{c_{34} - \sqrt{-\Delta}\sin A}{c_{34} + \sqrt{-\Delta}\sin A}.$$

The integration is taken over path from (A, B, C, D, E, F) to (A_0, B, C, D, E, F) where A_0 is a root of $\Delta = 0$.

Symmetric polyhedra

A tetrahedron $T = T(A, B, C, D, E, F)$ is called to be *symmetric* if $A = D,\ B = E,\ C = F$.

Theorem 12 (Derevnin-Mednykh-Pashkevich, 2004)

Let T be a symmetric hyperbolic tetrahedron. Then $Vol(T)$ is given by

$$2\int_{\Theta}^{\pi/2} (\arcsin(a\cos t) + \arcsin(b\cos t) + \arcsin(c\cos t) - \arcsin(\cos t))\frac{dt}{\sin 2t},$$

where $a = \cos A,\ b = \cos B,\ c = \cos C,\ \Theta \in (0, \pi/2)$ *is defined by*

$$\frac{\sin A}{\sinh \ell_A} = \frac{\sin B}{\sinh \ell_B} = \frac{\sin C}{\sinh \ell_C} = \tan\Theta,$$

and ℓ_A, ℓ_B, ℓ_C *are the lengths of the edges of* T *with dihedral angles* $A, B, C,$ *respectively.*

- Sine and cosine rules for hyperbolic tetrahedron

Let $T = T(A, B, C, D, E, F)$ be a hyperbolic tetrahedron with dihedral angles A, B, C, D, E, F and edge lengths $\ell_A, \ell_B, \ell_C, \ell_D, \ell_E, \ell_F$ respectively.

Consider two Gram matrices

$$G = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

and

$$G^* = \begin{pmatrix} 1 & \cosh \ell_D & \cosh \ell_E & \cosh \ell_C \\ \cosh \ell_D & 1 & \cosh \ell_F & \cosh \ell_B \\ \cosh \ell_E & \cosh \ell_F & 1 & \cosh \ell_A \\ \cosh \ell_C & \cosh \ell_B & \cosh \ell_A & 1 \end{pmatrix}.$$

Starting volume calculation for tetrahedra we rediscover the following classical result:

Theorem 13 (Sine Rule, E. d'Ovidio, 1877, J. L. Coolidge, 1909, W. Fenchel, 1989)

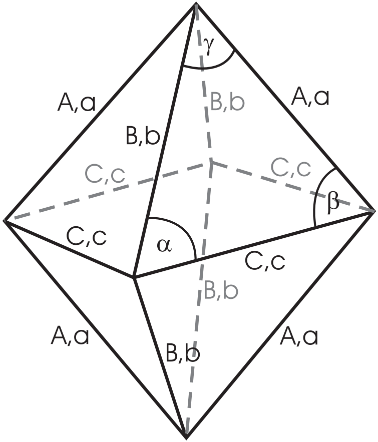
$$\frac{\sin A \sin D}{\sinh \ell_A \sinh \ell_D} = \frac{\sin B \sin E}{\sinh \ell_B \sinh \ell_E} = \frac{\sin C \sin F}{\sinh \ell_C \sinh \ell_F} = \sqrt{\frac{\det G}{\det G^*}}.$$

The following result seems to be new or at least well-forgotten.

Theorem 14 (Cosine Rule, M. Pashkevich and M., 2005)

$$\frac{\cos A \cos D - \cos B \cos E}{\cosh \ell_B \cosh \ell_E - \cosh \ell_A \cosh \ell_D} = \sqrt{\frac{\det G}{\det G^*}}.$$

Both results are obtained as a consequence of Theorem 11 relating complimentary minors of matrices G and G^*.



Octahedron $\mathcal{O} = \mathcal{O}(a, b, c, A, B, C)$ having $mmm-$symmetry

Theorem 15 (Sine-Tangent Rule, N. Abrosimov, M. Godoy and M., 2008)

Let $\mathcal{O}(a, b, c, A, B, C)$ be a spherical octahedra having mmm-symmetry. Then the following identities hold

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin C}{\tan c} = T = 2\,\frac{K}{\mathcal{C}}\,,$$

where K and $\mathcal{C}$ are positive numbers defined by the equations

$$K^2 = (z - xy)(x - yz)(y - xz), \quad \mathcal{C} = 2\,xyz - x^2 - y^2 - z^2 + 1\,,$$

and $x = \cos a$, $y = \cos b$, $z = \cos c$.

Symmetric polyhedra: volume of *mmm*− octahedron

Theorem 16 (N. Abrosimov, M. Godoy and M., 2008)

Let $\mathcal{O} = \mathcal{O}(A, B, C)$ *be a spherical octahedron having* mmm*–symmetry. Then volume* $V = V(\mathcal{O})$ *is given*

$$2\int_{\frac{\pi}{2}}^{\theta}\Big(\mathrm{arth}(X\cos\tau) + \mathrm{arth}(Y\cos\tau) + \mathrm{arth}(Z\cos\tau) + \mathrm{arth}(\cos\tau)\Big)\frac{d\tau}{\cos\tau},$$

where $X = \cos A$, $Y = \cos B$, $Z = \cos C$ *and* $0 \le \theta \le \pi/2$ *is a root of the equation*

$$\tan^2\theta + \frac{(1+X)(1+Y)(1+Z)}{1+X+Y+Z} = 0.$$

Moreover, θ *is given by the Sine−Tangent rule*

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin C}{\tan c} = \tan\theta\,.$$

Symmetric polyhedra: Euclidean $mmm-$ octahedron

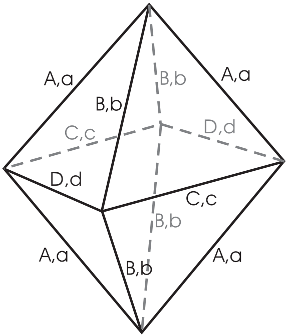
For the Euclidean case the following result holds.

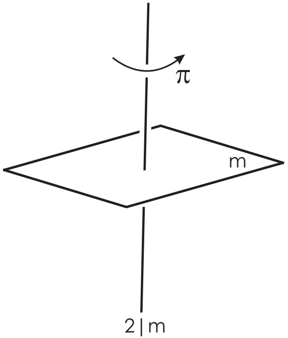
Theorem 17 (R. V. Galiulin, S. N. Mikhalev, I. Kh. Sabitov, 2004)

Let V be the volume of an Euclidean octahedron $\mathcal{O}(a, b, c, A, B, C)$ with mmm–symmetry. Then V is a positive root of equation

$$9V^2 = 2(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)(b^2 + c^2 - a^2)\,.$$

Symmetric polyhedra: octahedron with $2|m$–symmetry





Octahedron $\mathcal{O} = \mathcal{O}(a, b, c, d, A, B, C, D)$, having $2|m$–symmetry.

Symmetric polyhedra: volume of $2|m$–octahedron

Theorem 18 (N. Abrosimov, M. Godoy and M., 2008)

Let $\mathcal{O} = \mathcal{O}(A, B, C, D)$ *be a spherical octahedron having* $2|m$ *–symmetry. Then the volume* $V = V(\mathcal{O})$ *is given by*

$$2\int\limits_{\frac{\pi}{2}}^{\theta}\Big(\mathrm{arth}(X\cos\tau) + \mathrm{arth}(Y\cos\tau) + \mathrm{arth}(Z\cos\tau) + \mathrm{arth}(W\cos\tau)\Big)\frac{d\tau}{\cos\tau},$$

where $X = \cos A,\ Y = \cos B,\ Z = \cos\frac{C+D}{2},\ W = \cos\frac{C-D}{2}$ *and*

$\theta,\ 0 \le \theta \le \pi/2$ *is given by Sine–Tangent rule*

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin\frac{C+D}{2}}{\tan\frac{c+d}{2}} = \frac{\sin\frac{C-D}{2}}{\tan\frac{c-d}{2}} = \tan\theta\,.$$

Symmetric polyhedra: Euclidean $2|m$–octahedron

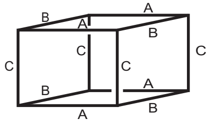
For the Euclidean case the following result holds.

Theorem 19 (R. V. Galiulin, S. N. Mikhalev, I. Kh. Sabitov, 2004)

Let V be the volume of an Euclidean octahedron $\mathcal{O}(a, b, c, d, A, B, C, D)$ with $2|m$–symmetry. Then V is a positive root of equation

$$9V^2 = (2a^2 + 2b^2 - c^2 - d^2)(a^2 - b^2 + cd)(b^2 - a^2 + cd)\,.$$

Symmetric polyhedra: volume of spherical hexahedron



Hexahedron $\equiv$ combinatorial cube $\mathcal{H}(A, B, C)$

Theorem 20 (N. Abrosimov, M. Godoy and M., 2008)

Volume of a spherical hexahedron $\mathcal{H}(A, B, C)$ *with* $mmm-$ *symmetry is equal*

$$2Re\int\limits_{\frac{\pi}{2}}^{\Theta}(arcth(\frac{X}{\cos t}) + arcth(\frac{Y}{\cos t}) + arcth(\frac{Z}{\cos t}) + arcth(\frac{1}{\cos t}))\frac{dt}{\sin t},$$

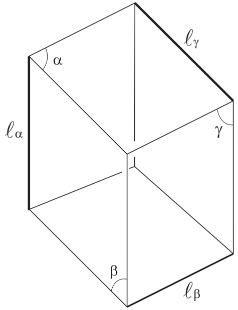
where $\Theta,\ 0 \le \Theta \le \frac{\pi}{2}$ *is defined by*

$$\tan^2\Theta + \frac{(2XYZ + X^2 + Y^2 + Z^2 - 1)^2}{4(X + YZ)(Y + XZ)(Z + XY)} = 0,$$

$X = \cos A,\ \ Y = \cos B$ *and* $Z = \cos C.$

Lambert cube

The Lambert cube $Q(\alpha, \beta, \gamma)$ is one of the simplest polyhedra. By definition, this is a combinatorial cube with dihedral angles α, β and γ at three noncoplanar edges and with right angles at all other edges. The volume of the Lambert cube in hyperbolic space was obtained by R. Kellerhals (1989) in terms of the Lobachevsky function. We give the volume formula of the Lambert cube in spherical space.



Lambert cube: spherical volume

Theorem 21 (D. A. Derevnin and M., 2009)

The volume of a spherical Lambert cube $Q(\alpha, \beta, \gamma)$, $\frac{\pi}{2} < \alpha, \beta, \gamma < \pi$ *is given by the formula*

$$V(\alpha, \beta, \gamma) = \frac{1}{4}(\delta(\alpha, \Theta) + \delta(\beta, \Theta) + \delta(\gamma, \Theta) - 2\delta(\frac{\pi}{2}, \Theta) - \delta(0, \Theta)),$$

where

$$\delta(\alpha, \Theta) = \int\limits_{\Theta}^{\frac{\pi}{2}} \log(1 - \cos 2\alpha \cos 2\tau) \frac{d\tau}{\cos 2\tau}$$

and Θ, $\frac{\pi}{2} < \Theta < \pi$ *is defined by*

$$\tan^2 \Theta = -K + \sqrt{K^2 + L^2 M^2 N^2}, \ K = (L^2 + M^2 + N^2 + 1)/2,$$

$L = \tan \alpha, \ M = \tan \beta, \ N = \tan \gamma.$

Remark. The function $\delta(\alpha, \Theta)$ can be considered as a spherical analog of the function

$$\Delta(\alpha, \Theta) = \Lambda(\alpha + \Theta) - \Lambda(\alpha - \Theta).$$

Then the main result of R.Kellerhals (1989) for hyperbolic volume can be obtained from the above theorem by replacing $\delta(\alpha, \Theta)$ to $\Delta(\alpha, \Theta)$ and K to $-K$.

Lambert cube: hyperbolic volume

As a consequence of the above mentioned volume formula for Lambert cube we obtain

Proposition 1 (D. A. Derevnin and M., 2009)

Let $L(\alpha, \beta, \gamma)$ *be a spherical Lambert cube such that* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. *Then*

$$Vol\, L(\alpha, \beta, \gamma) = \frac{1}{4}\left(\frac{\pi^2}{2} - (\pi - \alpha)^2 - (\pi - \beta)^2 - (\pi - \gamma)^2\right).$$

Before a similar statement for spherical orthoscheme was done by Coxeter.

Proposition 2 (H. S. M. Coxeter, 1935)

Let $T(\alpha, \beta, \gamma)$ *be a spherical orthoscheme such that* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. *Then*

$$Vol\, T(\alpha, \beta, \gamma) = \frac{1}{4}\left(\beta^2 - \left(\frac{\pi}{2} - \alpha\right)^2 - \left(\frac{\pi}{2} - \gamma\right)^2\right).$$

Rational Volume Problem

The following problem is widely known and still open.

Rational Volume Problem. Let P be a spherical polyhedron whose dihedral angles are in $\pi\mathbb{Q}$. Then $Vol\,(P) \in \pi^2\mathbb{Q}$.

- Examples

1. Since $\cos^2\frac{2\pi}{3} + \cos^2\frac{2\pi}{3} + \cos^2\frac{3\pi}{4} = 1$, by Proposition 1 we have

$$Vol\,L\,(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}) = \frac{31}{576}\pi^2.$$

2. Let P be a Coxeter polyhedron in S^3 (that is all dihedral angles of P are $\frac{\pi}{n}$ for some $n \in \mathbb{N}$). Then the Coxeter group $\Delta(P)$ generated by reflections in faces of P is finite and

$$Vol\,(P) = \frac{Vol\,(S^3)}{|\Delta(P)|} = \frac{2\pi^2}{|\Delta(P)|} \in \pi^2\mathbb{Q}.$$