Packings of Equal Circles on Flat Tori

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Workshop on Rigidity
Fields Institute
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Introduction

Goal

Understand locally and globally maximally dense packings of equal circles on a fixed torus.
<table>
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<th>Which Torus?</th>
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<td>A flat torus is the quotient of the plane by a rank 2 lattice, $\mathbb{R}^2/\Lambda$</td>
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Which Torus?

A flat torus is the quotient of the plane by a rank 2 lattice, $\mathbb{R}^2/\Lambda$

The action of $SL(2,\mathbb{Z})$ (and scaling) on oriented lattices preserves the density of a packing and can be used to put the lattice into a normal form. As we are working with unoriented lattices this is further reduced to the following forms:
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For the optimal packings of 2 circles on any torus with a length one closed geodesic see the work of Przeworski (2006).
A flat torus is the quotient of the plane by a rank 2 lattice, $\mathbb{R}^2/\Lambda$

The action of $SL(2,\mathbb{Z})$ (and scaling) on oriented lattices preserves the density of a packing and can be used to put the lattice into a normal form. As we are working with unoriented lattices this is further reduced to the following forms:

A Square Torus is the quotient of the plane by unit perpendicular vectors. See the work of H. Mellisen (1997) – proofs for 3 and 4 circles and conjectures up to 19 circles. For large numbers (> 50) see the work of Lubachevsky, Graham, and Stillinger (1997).
Introduction

Which Torus?

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The action of $SL(2, \mathbb{Z})$ (and scaling) on oriented lattices preserves the density of a packing and can be used to put the lattice into a normal form. As we are working with unoriented lattices this is further reduced to the following forms:

A Rectangular Torus is the quotient of the plane by perpendicular vectors. See the work of A. Heppes (1999) – proofs for 3 and 4 circles.
A flat torus is the quotient of the plane by a rank 2 lattice, $\mathbb{R}^2/\Lambda$.

The action of $SL(2, \mathbb{Z})$ (and scaling) on oriented lattices preserves the density of a packing and can be used to put the lattice into a normal form. As we are working with unoriented lattices this is further reduced to the following forms:

A Triangular Torus is the quotient of the plane by unit vectors with a 60 degree angle between them. Understanding packings on this torus might help prove a conjecture of L. Fejes Tóth on the solidity of the triangular close packing in the plane with one circle removed.
Circle Packing
Packing Graphs & Strut Frameworks

Circle Packing ⇔ Equilateral Toroidal Packing Graph
Circle Packing ⇔ Equilateral Toroidal Packing Graph ⇒ Combinatorial Graph
Viewing the packing graph as a strut framework helps us understand the possible combinatorial (multi-)graphs.
Consider the optimal packing of seven circles a hard boundary square. Due to Schear/Graham (1965) Mellisen (1997).
Rigid Spine And Free Circles

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Rigid Spine And Free Circles

Consider the optimal packing of seven circles a hard boundary square. Due to Schear/Graham(1965) Mellisen(1997)

The red circle is a *free circle* and the packing graph associated to the green circles form the *rigid spine*. In what follows we will only consider packings without free circles.
An assignment of vectors \((\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots, \vec{p}_n)\) to each of the vertices \((p_1, p_2, p_3, \ldots, p_n)\) in a toroidal strut framework is a \textit{infinitesimal flex} of the arrangement if

\[(\vec{p}_i - \vec{p}_j) \cdot (p_i - p_j) \geq 0\]

for each strut \((i, j)\) in the framework.
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Notes:

- As this is a toroidal framework $(p_i - p_j)$ will depend on more than just the vertices. The homotopy class of the struts matters.
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- This forms a system of homogeneous linear inequalities.
Strut Frameworks: Rigidity and Infinitesimal Rigidity

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- As this is a toroidal framework \((p_i - p_j)\) will depend on more than just the vertices. The homotopy class of the struts matters.
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**Theorem (Connelly)**

A \textit{(toroidal) strut framework is (locally) rigid if and only if infinitesimally rigid}
Optimal Arrangements and Toroidal Strut Frameworks

Observation
Given a packing, if the associated toroidal strut framework is (locally) rigid then the packing is locally maximally dense.
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Theorem (Connelly)
If a toroidal packing is locally maximally dense then there is a subpacking whose associated toroidal strut framework is (locally) rigid.
Theorem (Connelly)

A locally maximally dense packing of $n$ circles on a flat torus (without free circles) has at least $2n - 1$ edges.

Observations:
Conbinatorial Graph Edge Restrictions

**Theorem (Connelly)**

A locally maximally dense packing of \(n\) circles on a flat torus (without free circles) has at least \(2n - 1\) edges.

**Observations:**
- Each circle is tangent to at most 6 others
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  - A combinatorial graph associated to an optimal packing has between $2n - 1$ and $3n$ edges.
Combinatorial Graph Edge Restrictions

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*A locally maximally dense packing of $n$ circles on a flat torus (without free circles) has at least $2n - 1$ edges.*

Observations:

- Each circle is tangent to at most 6 others
  - → A combinatorial graph associated to an optimal packing has between $2n - 1$ and $3n$ edges.
- To be infinitesimally rigid each circle must be tangent to at least 3 others and the points of tangency can’t be restricted to a closed semi-circle.
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Note: These observations are enough to determine all the optimal packings of 1-4 circles on a square flat torus.
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   - Alternatively, construct the equilateral embedding and let it determine the torus or tori onto which it embeds.

4. Determine which equilateral embeddings are associated to locally maximally dense packings.
Three Circle Case

Step 1: Partial list of possible combinatorial graphs.
Step 2: Partial list of all embeddings of the combinatorial graphs on a topological torus.

Many embeddings with all circles self tangent
Three Circle Case

Steps 3 & 4: Equilateral Embeddings and Locally Maximally Dense Packing/Regions.
Minimally Dense Arrangements

Rectangular Torus, \( \approx 1.35 \) ratio

Density \( = \frac{2\pi \sqrt{3}}{\sqrt{138+22\sqrt{33}}} \approx 0.66930 \)
Minimally Dense Arrangements

Equilateral Torus, 100 Degrees
Density $= \frac{3\pi}{16 \sin\left(\frac{4\pi}{9}\right)} \approx 0.61673$

Rectangular Torus, $\approx 1.35$ ratio
Density $= \frac{2\pi \sqrt{3}}{\sqrt{138+22\sqrt{33}}} \approx 0.66930$
Definition

A collection of scalars $\omega_{ij} = \omega_{ji}$ (one for each strut) is called an self-stress if $\sum_j \omega_{ij}(p_j - p_i) = \vec{0}$ for all vertices $p_i$. 


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Theorem (Roth-Whiteley)

A toroidal strut framework is (infinitesimally) rigid if and only if it is infinitesimally rigid as a bar framework and it has a self-stress that has the same sign and is non-zero on every strut.
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A collection of scalars $\omega_{ij} = \omega_{ji}$ (one for each strut) is called an self-stress if $\sum_j \omega_{ij}(p_j - p_i) = 0$ for all vertices $p_i$.

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Theorem (Connelly)

On a fixed torus, suppose there is a packing so that the associated equilateral strut framework, $F$, is infinitesimally rigid then any other infinitesimally rigid, equilateral strut framework freely homotopic to $F$ on the torus is congruent to $F$ by translation.
Two Locally Optimally Dense Arrangements on One Torus
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The packing graphs are not homotopic on the fixed torus.
Using the same techniques the following are optimally dense.

- 5 Circles
- Square Torus
- 10 contacts
Other Results on the Square and Triangular Torus

Using the same techniques the following are optimally dense.

5 Circles
Square Torus
10 contacts

5 Circles
Triangular Torus
9 contacts
Other Results on the Square and Triangular Torus

Using the same techniques the following are optimally dense.

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5 Circles
Triangular Torus
9 contacts

6 Circles
Triangular Torus
18 contacts
A packing on a torus is *strictly jammed* if there is no non-trivial infinitesimal motion of the packing, as well as the lattice defining the torus, subject to the condition that the total area of the torus does not infinitesimally increase.
Strictly Jammed / Periodically Stable Packings

Definition

A packing on a torus is *strictly jammed* if there is no non-trivial infinitesimal motion of the packing, as well as the lattice defining the torus, subject to the condition that the total area of the torus does not infinitesimally increase.

Counting for a toroidal packing of $n$ circles:
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Counting for a toroidal packing of $n$ circles:

**Constraints:**  Packing Edges: $e$  Area Constraint: 1
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Counting for a toroidal packing of $n$ circles:

**Constraints:** Packing Edges: $e$  Area Constraint: 1

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Counting for a toroidal packing of \( n \) circles:

\begin{itemize}
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  \item \textbf{Trivial Motions:} Translations: 2 \hspace{1cm} Rotation: 1
\end{itemize}
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A packing on a torus is *strictly jammed* if there is no non-trivial infinitesimal motion of the packing, as well as the lattice defining the torus, subject to the condition that the total area of the torus does not infinitesimally increase.

Counting for a toroidal packing of $n$ circles:

- **Constraints:** Packing Edges: $e$  Area Constraint: 1
- **Variables:** Coordinates: $2n$  Lattice Vectors: 4
- **Trivial Motions:** Translations: 2  Rotation: 1

To have a unique solution, you must have one more inequality/constraint than unconstrained variables so $(e + 1) \geq (2n + 4) - (2 + 1) + 1$ or

$$e \geq 2n + 1$$

in order to possibly be strictly jammed.
Non-Triangular-Close Based Strictly Jammed Example
(Connelly)

10 Circles
\[ \approx 75^\circ \] Torus with \( \approx 1.17 \) Ratio
22 contacts
Questions/Future Work

- Continue to explore packing of small numbers of circles on the torus or other smooth flat domains.
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- How can we algorithmically or computationally determine if an embedded toroidal graph
  - has an equilateral embedding
  - corresponds to a locally optimal packing
- Find other examples of strictly jammed packings and work toward understanding the connection between a packing being strictly jammed and packings such that every cover of the torus is locally optimal.
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- How can we algorithmically or computationally determine if an embedded toroidal graph
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- Is this algorithm practical for toroidal bi- or poly-dispersed packings?
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- How can we algorithmically or computationally determine if an embedded toroidal graph has an equilateral embedding and corresponds to a locally optimal packing.
- Find other examples of strictly jammed packings and work toward understanding the connection between a packing being strictly jammed and packings such that every cover of the torus is locally optimal.
- Is this algorithm practical for toroidal bi- or poly-dispersed packings?
- Is there a 3-d analog for this algorithm for packing sphere in a 3-torus?
Thank You

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