Conductivity imaging from one interior measurement

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A convergent algorithm to solve

\[ u = \arg\min \left\{ \int_{\Omega} |J| |\nabla v| : v \in H^1(\Omega), \ v|_{\partial\Omega} = f \right\}. \]

Joint work with A. Nachman and A. Timonov
Let $u_f \in H^1(\Omega)$ with $u_f |_\Omega = f$. Then our weighted minimization problem can be written as

$$(P) \quad \inf_{v \in H^1_0(\Omega)} \int_\Omega |J| |\nabla v + \nabla u_f|.$$

The dual problem is

$$(D) \quad \sup \{ \langle \nabla u_f, b \rangle : b \in (L^2(\Omega))^n, \ |b(x)| \leq |J(x)| \ a.e. \ and \ \nabla \cdot b \equiv 0 \}.$$
Theorem (M, A. Nachman, A. Timonov (2011))

Assume that the data \((|J|, f)\) is admissible. Then

\[
\inf_{v \in H^1_0(\Omega)} \int_{\Omega} |J||\nabla v + \nabla u_f|
\]

\[
= \sup\{<\nabla u_f, b>: b \in (L^2(\Omega))^n, |b(x)| \leq |J(x)| \text{ a.e. and } \nabla \cdot b \equiv 0\}
\]

and the current density \(J\) corresponding to the voltage potential \(f\) on \(\partial \Omega\) is the unique solution of the dual problem.
Let \( E : (L^2(\Omega))^n \to \mathbb{R} \) and \( G : H^1_0(\Omega) \to \mathbb{R} \) be defined by

\[
E(d) = \int_\Omega |J||d + \nabla u_f| \quad \text{and} \quad G(\nu) \equiv 0.
\]

Then the dual problem can be written in the form

\[
(D) \quad \min_{b \in (L^2(\Omega))^n} \{ E^*(b) + G^*(-\nabla \cdot b) \}.
\]

Since \( J \) is the solution of the dual problem

\[
0 \in \partial E^*(J) + \partial [G^* o (-\nabla \cdot )](J).
\]

Let \( A := \partial E^*(J) \) and \( B := \partial [G^* o (-\nabla \cdot )] \). Then above can be written as

\[
0 \in A(J) + B(J),
\]

where \( A \) and \( B \) are maximal monotone set-valued operators.
To solve

\[ 0 \in A(J) + B(J) \]

we apply a Douglas-Rachford algorithm. This algorithm produces two sequences \( p_k \) and \( x_k \) such that

\[ p_k \rightharpoonup J \quad \text{and} \quad x_k \rightharpoonup \nabla u. \]
Theorem (Lions and Mercier (1979), Svaiter (2010))

Let $H$ be a Hilbert space and $A, B$ be maximal monotone operators and assume that a solution of (1) exists. Then, for any initial elements $x_0$ and $p_0$ the sequences $p_k$ and $x_k$ generated by the following algorithm

\[
x_{k+1} = R_A(2p_k - x_k) + x_k - p_k
\]
\[
p_{k+1} = R_B(x_{k+1}),
\]

converges weakly to some $\hat{x}$ and $\hat{p}$ respectively. Furthermore, $\hat{p} = R_B(\hat{x})$ and $\hat{p}$ satisfies

\[
0 \in A(\hat{p}) + B(\hat{p}).
\]  

(1)

\[
R_A = (Id + A)^{-1}
\]
Let \( u_f \in H^1(\Omega) \) with \( u_f|_{\partial \Omega} = f \), and initialize \( b^0, d^0 \in (L^2(\Omega))^n \). For \( k \geq 1 \):

1. Solve
   \[
   \Delta u^{k+1} = \nabla \cdot (d^k(x) - b^k(x)), \quad u^{k+1}|_{\partial \Omega} = f.
   \]

2. Compute
   \[
   d^{k+1} := \begin{cases} 
   \max\{ |\nabla u^{k+1} + b^k| - |J|, 0 \} \frac{\nabla u^{k+1} + b^k}{|\nabla u^{k+1} + b^k|} & \text{if } |\nabla u^{k+1}(x) + b^k(x)| \neq 0, \\
   0 & \text{if } |\nabla u^{k+1}(x) + b^k(x)| = 0.
   \end{cases}
   \]

3. Let
   \[
   b^{k+1}(x) = b^k(x) + \nabla u^{k+1}(x) - d^{k+1}(x).
   \]

This is an alternating split Bregman algorithm of Goldstein and Osher applied to the primal problem (P).
Theorem (M, A. Nachman, A. Timonov (2011))

The sequences $b^k$, $d^k$, and $u^k$ produced by the above algorithm converge weakly to $J$, $\nabla u$, and $u$, respectively.

So we are simultaneously solving the primal and the dual problem.
Numerical simulations
To simulate the internal data $|J|$ we use a CT (Computed Tomography) image of human abdomen rescaled to a realistic range of tissue conductivities.

![Original image](left) and reconstructed image with 60 iterations (right).

**Figure:** Original image (left) and reconstructed image with 60 iterations (right).
Figure: Conductivity reconstruction with the boundary condition $f(x, y) = y$ for $N = 1, 5, 10, 30, 50, 100$ iterations.
Figure: Magnitude of the current density $|J|$ for the non two-to-one boundary data $f(x, y) = y + 2\sin(7\pi y)$. 
Figure: Conductivities constructed using the alternating split Bregman algorithm with $N = 1, 5, 10, 30, 50, 100$ iterates for the non two-to-one boundary data $f(x, y) = y + 2\sin(7\pi y)$. 
Numerical errors for 100 iterations.

<table>
<thead>
<tr>
<th>Low Noise (Level=0.01)</th>
<th>Moderate Noise (Level=0.035)</th>
<th>Higher Noise (Level=0.06)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.026</td>
<td>0.080</td>
<td>0.152</td>
</tr>
</tbody>
</table>

Figure: Low noise (left), moderate noise (middle), and higher noise (right).
**Figure:** Reconstruction in the presence of the perfectly conducting (right) and insulating (left) inclusions.