An Introduction to Quasi-Symmetric and Noncommutative Symmetric Functions.

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Warning and Outline

This talk has nothing to do with k-Schur functions, affine Grassmanians or any other topic of this school...
1. (Another) Tale of Two Algebras: Motivation

2. Notations, conventions, etc.
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2. Notations, conventions, etc.
3. Quasi-Symmetric Functions.
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(Another) Tale of Two Algebras:
Motivation
Notations, conventions, etc.
Quasi-Symmetric Functions.
Noncommutative Symmetric Functions.

Magic Triangle

```
NSym ← − − − − QSym
                      ↑↑↑↑↑↑↑↑↑
                           ←− ←−
                           →→
                           ↑
                           →
                           ↓
 Sym
```

Sym:
\[ m^{\lambda}, h^{\lambda}, s^{\lambda}, ... \]

NSym:
\[ M^I, L^I, S^I, R^I, ... \]

QSym:
\[ M^I, L^I, ... \]
(Another) Tale of Two Algebras: Motivation

Notations, conventions, etc.

Quasi-Symmetric Functions.

Noncommutative Symmetric Functions.

Magic Triangle

\[ \text{NSym} \leftarrow \text{Sym} \rightarrow \text{QSym} \]

**Sym**:
- \( m_{\lambda} \), \( h_{\lambda} \), \( s_{\lambda} \), ...

**NSym**:
- \( M^I \), \( L^I \), \( S^I \), \( R^I \), ...

**QSym**:
- \( M_I \), \( L_I \), ...
A composition is ordered set of integers: $l = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|l|$, and the number of parts – by $\ell(l)$.

\[
l = (3, 1, 1, 4, 2), \quad |l| = 11, \quad \ell(l) = 5
\]
A composition is ordered set of integers: \( I = (i_1, \ldots, i_n) \). The sum of all parts is denoted by \(|I|\), and the number of parts – by \( \ell(I) \).

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$$I = (3, 1, 1, 4, 2), \quad |I| = 11, \quad \ell(I) = 5$$
Reverse refinement order.

Let \( I = (i_1, \ldots, i_n) \), \( J = (j_1, \ldots, j_k) \), \(|J| = |I|\) then \( I \) is greater in the reverse refinement order (or, simply, finer) than \( J \),

\[ I \succ J \]

if every part of \( J \) can be obtained by summing some consecutive parts of \( I \):

\[ J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n) \]
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$$J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$$

For instance, $(3, 3, 2) = (3, 1 + 2, 2) \prec (3, 1, 2, 2)$,
Reverse refinement order.

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Noncommutative Operations on Compositions

For two compositions $I = (i_1, \ldots, i_{r-1}, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ one defines two operations

$$ I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s) $$
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and

$$I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s)$$
Noncommutative Operations on Compositions

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Descent Sets and Compositions.

Another way to encode a composition \( I \) of \( n \) is by a subset \( D \) of \( \{1, 2, \ldots, n - 1\} \). If \( D = \{d_1, d_2, \ldots, d_k\} \), then

\[
I = (d_1, d_2 - d_1, d_3 - d_2, \ldots, n - d_k)
\]

Example: Let \( n = 6 \) and take a set \( \{2, 3, 5\} \)
Descent Sets and Compositions.

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Example: Let $n = 6$ and take a set $\{2, 3, 5\}$

![Diagram of descent sets and compositions]
Definitions of Quasi-Symmetric Functions.

For every composition $l = (i_1, \ldots, i_k)$, the quasi-symmetric monomial is defined

$$M_l = \sum_{s_1 < \ldots < s_k} x_{s_1}^{i_1} \ldots x_{s_k}^{i_k}$$
Definitions of Quasi-Symmetric Functions.

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$$M_I = \sum_{s_1 < \ldots < s_k} x_{s_1}^{i_1} \ldots x_{s_k}^{i_k}$$

and **quasi-symmetric fundamental**

$$L_I = \sum_{J \geq I} M_J$$
Examples of Quasi-Symmetric Functions

Monomials:

\[ M_{12}(x_1, x_2, x_3) = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 \]

\[ M_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 \]
Examples of Quasi-Symmetric Functions

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So that

\[ m_{21} = M_{21} + M_{12} \]
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In general,

\[ m_\lambda = \sum_{I: \mathcal{S}(I)=\lambda} M_I \]
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In general,

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Fundamental:

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Examples of Quasi-Symmetric Functions

Monomials:

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So that

\[ m_{21} = M_{21} + M_{12} \]

In general,

\[ m_\lambda = \sum_{I: \mathcal{S}(I)=\lambda} M_I \]

Fundamental:

\[ L_{12} = M_{12} + M_{1^3} \]
\[ L_{13} = M_{13} + M_{1^22} + M_{121} + M_{1^4} \]
Expansion of Schur Functions in Quasi-Symmetric Fundamental.

Consider a standard (skew-)tableau. A **descent** of SYT $T$ is an integer $i$ such that $i+1$ appears in a row of $T$ above $i$. The descent set of $T$, $\text{Des}(T)$ – is the set of all descents of $T$. Example: (descents are marked in bold)

$$
\begin{array}{cccc}
1 & 4 & & \\
2 & 3 & & \\
\end{array} & \begin{array}{cccc}
2 & 4 & & \\
1 & 3 & & \\
\end{array} & \begin{array}{cccc}
2 & 3 & & \\
1 & 4 & & \\
\end{array} & \begin{array}{cccc}
3 & 4 & & \\
1 & 2 & & \\
\end{array} & \begin{array}{cccc}
1 & 3 & & \\
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\end{array}
$$
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\end{array} \quad \begin{array}{cccc}
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\end{array} \quad \begin{array}{cccc}
2 & 3 & & \\
1 & 4 & & \\
\end{array} \quad \begin{array}{cccc}
3 & 4 & & \\
1 & 2 & & \\
\end{array} \quad \begin{array}{cccc}
1 & 3 & & \\
2 & 4 & & \\
\end{array}
\]

\[
\sum_{T: \text{SYT of shape } \lambda/\mu} L_{\text{Des}(T)}
\]

$L$
Expansion of Schur Functions in Quasi-Symmetric Fundamental.

Consider a standard (skew-)tableau. A **descent** of SYT $T$ is an integer $i$ such that $i + 1$ appears in a row of $T$ above $i$. The descent set of $T$, $\text{Des}(T)$ – is the set of all descents of $T$.

Example: (descents are marked in bold)

\[
\begin{align*}
1 & 4 & 2 & 3 \\
2 & 4 & 1 & 3 \\
2 & 3 & 1 & 4 \\
3 & 4 & 1 & 2 \\
1 & 3 & 2 & 4
\end{align*}
\]

\[
s_{\lambda/\mu} = \sum_{T: \text{SYT of shape } \lambda/\mu} L_{\text{Des}(T)}
\]

Example continues:

\[
s_{32/1} = L_{3,1} + L_{1,2,1} + L_{1,3} + 2L_{2,2}
\]
The **backsteps** of a permutation $w = (w_1, w_2, \ldots, w_n) \in S_n$ are $BS(w) = \{i \mid i + 1 \text{ is to the left of } i \text{ in } w\}$. Denote the reading word (left to right, top to bottom) of $T - w(T)$. Then

$$Des(T) = BS(w(T))$$
The **backsteps** of a permutation $w = (w_1, w_2, \ldots, w_n) \in S_n$ are $BS(w) = \{i \mid i + 1 \text{ is to the left of } i \text{ in } w\}$. Denote the reading word (left to right, top to bottom) of $T - w(T)$. Then

$$Des(T) = BS(w(T))$$

So, equivalently, one can look at the reading words of these tableaux: 1423, 2413, 2314, 3412, 1324 and record their backsteps.

$$s_{\lambda/\mu} = \sum_{T: \text{SYT of shape } \lambda/\mu} L_{BS(w(T))}$$
Recall that in $\text{Sym}$ there is a number of identities expressing one type of function (elementary, complete, Schur) as a determinant of other (power sums, complete, etc.). For instance,

$$e_n = \frac{1}{n!} \begin{vmatrix} p_1 & 1 & \ldots & 0 & 0 \\ p_2 & p_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-1} & \ldots & \ldots & p_1 & n-1 \\ p_n & \ldots & \ldots & p_2 & p_1 \end{vmatrix}$$
Quasi-Determinants.

Consider an almost-triangular matrix with noncommutative entries $a_{ij}$ and commutative off-diagonal entries $b_j$. Its quasideterminant (with respect to the bottom left element) is a sum of all weighted paths starting at the bottom row, ending at the first column, taking northward until encountering commutative off-diagonal entry and then turning east.

$$\begin{vmatrix}
a_{11} & b_1 & 0 \\
a_{21} & a_{22} & b_2 \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{31} - \frac{a_{32}a_{11}}{b_1} - \frac{a_{33}a_{21}}{b_2} + \frac{a_{33}a_{22}a_{11}}{b_1b_2}$$
Define **elementary symmetric** functions $\Lambda_n$:

$$\Lambda_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \psi_1 & 1 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n-1} & \psi_{n-2} & \ldots & \ldots & n - 1 \\ \psi_n & \psi_{n-1} & \ldots & \ldots & \psi_1 \end{vmatrix}$$
Noncommutative Elementary and Homogeneous Symmetric Functions.

Define **elementary symmetric** functions $\Lambda_n$:

$$\Lambda_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \psi_1 & 1 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n-1} & \psi_{n-2} & \ldots & \ldots & n-1 \\ \psi_n & \psi_{n-1} & \ldots & \ldots & \psi_1 \end{vmatrix}$$

and **complete symmetric** functions $S_n$:

$$S_n = \frac{1}{n} \begin{vmatrix} \psi_1 & -(n-1) & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n-1} & \psi_{n-2} & \ldots & \ldots & -1 \\ \psi_n & \psi_{n-1} & \ldots & \ldots & \psi_1 \end{vmatrix}$$
Noncommutative Monomials.

Define **noncommutative monomial symmetric function** corresponding to a composition \( l = (i_1, \ldots, i_n) \) as a quasideterminant of an \( n \) by \( n \) matrix:

\[
M^l = \frac{(-1)^{n-1}}{n} \begin{vmatrix}
\psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\
\psi_{i_{n-1}+i_n} & \psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
\psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \psi_{i_2} & n-1 \\
\psi_{i_1+\ldots+i_n} & \ldots & \ldots & \ldots & \psi_{i_1+i_2} & \psi_{i_1}
\end{vmatrix}
\]

where \( n = \ell(l) \).
Define noncommutative monomial symmetric function corresponding to a composition \( I = (i_1, \ldots, i_n) \) as a quasideterminant of an \( n \) by \( n \) matrix:

\[
M' = \frac{(-1)^{n-1}}{n} \begin{vmatrix}
\Psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\
\Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n - 1 \\
\Psi_{i_1+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1}
\end{vmatrix}
\]

where \( n = \ell(I) \). In particular

\[
M^{1^n} = \Lambda_n
\]

where \( \Lambda_n \) is an elementary symmetric function.
Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as a quasideterminant of an $n$ by $n$ matrix:

$$M^I = \frac{(-1)^{n-1}}{n} \left| \begin{array}{cccccc}
\Psi_{i_1} & 1 & 0 & \ldots & 0 & 0 \\
\Psi_{i_n} & \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Psi_{i_2+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\
\Psi_{i_1+i_n} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \\
\end{array} \right|$$

where $n = \ell(I)$. If one were to allow power sums to commute, say $\chi(\Psi_k) = p_k$, $\forall k$, i.e. projecting from $\text{NSym}$ to $\text{Sym}$, then

$$m_\lambda = \sum_{I: \mathcal{S}(I) = \lambda} \chi(M^I)$$
Noncommutative Fundamental and Ribbon Schur Functions.

Define **noncommutative fundamental** symmetric functions mimicking the definition in **QSym**

\[ L^I = \sum_{J \geq I} M^J \]

and **ribbon Schur functions** by Jacobi-Trudi formula using quasi-determinants:

\[ R^I = (-1)^{\ell(I)-1} \]

\[
\begin{array}{cccccc}
S_{i_n} & 1 & 0 & \ldots & \ldots \\
S_{i_n+i_{n-1}} & S_{i_{n-1}} & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
S_{i_n+\ldots+i_2} & S_{i_{n-1}+\ldots+i_2} & \ldots & S_{i_2} & 1 \\
S_{i_n+\ldots+i_1} & S_{i_{n-1}+\ldots+i_1} & \ldots & \ldots & S_{i_1}
\end{array}
\]
Genocchi Backsteps.

The **G-backsteps** of a permutation \( w = (w_1, w_2, \ldots, w_n) \in S_n \) are positions of \( GBS(w) = \{ i \mid i + 1 \text{ is to the left of } i \text{ in } w \} \) minus 1.

Example:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 3 & 4 & 1 \\
1 & 3 & 2 & 4 & 3 & 4 & 1 \\
2 & 1 & 4 & 3 & 2 & 1 & 4 \\
3 & 4 & 1 & 2 & 3 & 1 & 2 \\
\end{array}
\]

GBS(1423) = \{3\}
GBS(2413) = \{2, 3\}
GBS(2314) = \{2\}
GBS(3412) = \{3\}
GBS(1324) = \{2\}
Genocchi Backsteps.

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**Example**

\[
\begin{align*}
1 & 4 & 2 & 3 \\
2 & 4 & 1 & 3 \\
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\end{align*}
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GBS(1423) &= \{3\} \\
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GBS(2314) &= \{2\} \\
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GBS(1324) &= \{2\}
\end{align*}
\]
Expansion of Ribbon Schur in Noncommutative Fundamental.

\[ R^I = \sum_{T: \text{SYT of shape } I} L^{GBS(w((T))} \]
Expansion of Ribbon Schur in Noncommutative Fundamental.

\[ R^I = \sum_{T: \text{SYT of shape } I} L_{GBS}(w(T)) \]

Example:

\[ R^{2,2} = 2L^{3,1} + L^{2,1,1} + 2L^{2,2} \]