Torsion, Rank and Integer Points on Elliptic Curves

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Overview

0. Introductory Remarks
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I. Torsion
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II. Rank
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I. Torsion

II. Rank

III. Integer Points
Generalities

An elliptic curve defined over $\mathbb{Q}$:
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An elliptic curve defined over \( \mathbb{Q} \):

\[
y^2 = x^3 + Ax + B,
\]

\(A, B \in \mathbb{Z}, \ x^3 + Ax + B\) has only simple roots.

(short Weierstrass model)
Other Models of Elliptic Curves

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \]
(general Weierstrass equation)
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\[ ax^2 - by^2 = c, dx^2 - ez^2 = f \]
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\[ x^2 + y^2 = c^2(1 + dx^2y^2) \] (Edwards Curves)

\[ F(x, y) = 0 \] (\( F = 0 \) is a curve of genus 1)
Primary Objects of Study

\[ E(\mathbb{Q}) = \{(x, y) \in (\mathbb{Q})^2; y^2 = x^3 + Ax + B\} \cup \{\infty\}, \]
the group of rational points on \( E \).
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\( T \) is the \textit{torsion subgroup} of \( E(\mathbb{Q}) \), consisting of the points on \( E \) of finite order, and \( r = \text{Rank}(E) \).

\[ E(\mathbb{Z}) = \{(x, y) \in \mathbb{Z}^2; F(x, y) = 0\}, \]
where \( F(x, y) = 0 \) is a curve of genus 1.
Effective Results

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**Integral Points:** finiteness, upper bounds, algorithm to compute all points, specific results for families of curves
I.1 Torsion - group structure
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Mazur’s Theorem
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Then $T$ has one of the following two forms

i. A cyclic group of order $N$ with $1 \leq N \leq 10$ or $N = 12$.

ii. The product of a cyclic group of order 2 and a cyclic group of order $2N$, with $1 \leq N \leq 4$. 
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Let $K$ be a quadratic field, and let $E$ be an elliptic curve defined over $K$. Let $\mathcal{T}$ denote the subgroup of $E(K)$ consisting of the points of finite order.

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Kamienny’s Theorem

Let $K$ be a quadratic field, and let $E$ be an elliptic curve defined over $K$. Let $T$ denote the subgroup of $E(K)$ consisting of the points of finite order. Then $T$ has one of the following forms

i. A cyclic group of order $N$ with $1 \leq N \leq 16$ or $N = 18$.

ii. The product of a cyclic group of order 2 and a cyclic group of order $2N$, with $1 \leq N \leq 6$.

iii. The product of a cyclic group of order 3 and a cyclic group of order $2N$, with $1 \leq N \leq 2$.

iv. The product of two cyclic groups of order 4.
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Let $K$ be a number field of degree $d > 1$, and let $E$ be an elliptic curve defined over $K$. Let $\mathcal{T}$ denote the subgroup of $E(K)$ consisting of the points of finite order.

If $\mathcal{T}$ contains a point of prime order $p$, then

$$p < d^3 d^2.$$
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Corollary Let $d$ be a positive integer. There is a real number $B(d)$ with the property that for any elliptic curves $E$, defined over any number field $K$ of degree $d$, every torsion point in $E(K)$ has order bounded by $B(d)$. 
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Let $E$ be the elliptic curve defined by

$$y^2 = f(x) = x^3 + ax^2 + bx + c,$$

where $f(x)$ is a nonsingular cubic curve with integer coefficients $a, b, c$, and let

$$D = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2$$

represent the discriminant of $f$. 
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If $P = (x, y)$ is a point of finite order on $E$, then $x$ and $y$ are integers, and either

i. $y = 0$ (in which case $P$ has order 2), or

ii. $y$ divides $D$. (in fact $y^2$ divides $D$)
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This is an extremely useful computational tool.
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• Finally, determine cyclicity of the case $|T| = 4k$ by

  $$T = C_{4k} \text{ iff } f(x) = 0 \text{ has 3 integer roots}$$
  $$T = C_2 \times C_{2k} \text{ iff } f(x) = 0 \text{ has 1 integer root.}$$
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\[ D(E) = 27, \text{ and so for } (x, y) \in T(E), \text{ N-L implies } y \in \{0, \pm 1, \pm 3\}, \text{ and} \]
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\( D(E) = 27 \), and so for \((x,y) \in T(E)\), N-L implies \( y \in \{0, \pm 1, \pm 3\} \), and

\( T(E) \subseteq \{\infty, (-1, 0), (0, 1), (0, -1), (2, 3), (2, -3)\} \).
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\[ T(E) \cong C_6. \]
A Family of Curves

\[ E_k : y^2 = x^3 + k, \quad p^6 \parallel k \]
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All nontrivial torsion points are as follows:

1. If \( k = C^2 \), then \((0, \pm C)\) are of order 3.
2. If \( k = D^3 \), then \((-D, 0)\) is of order 2.
3. If \( k = 1 \), then \((2, \pm 3)\) are of order 6.
4. If \( k = -432 \), then \((12, \pm 36)\) are of order 3.
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4. If \( k = -432 \), then \((12, \pm 36)\) are of order 3.

Proof: First observe that \( x_{2P} = (w - 2)x_P \), where \( w = 9x_P^3/4y_P^2 \). Then use the Nagell-Lutz theorem to show that \( w \in \mathbb{Z} \), and that for \( |w - 2| > 1 \), \( P \) cannot have odd order.
Another Family of Curves

\[ E_A : y^2 = x^3 + Ax, \quad p^4 \parallel A \]
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Remark. \( E_A \) is related to Diophantine equations of the form \( u^2 - dy^4 = k \) with \( A = kd \).
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The nontrivial torsion points on \( E_A \) are:

1. \((0,0)\) is a point of order 2.
2. If \( A = 4 \), then \((2, \pm 4)\) are of order 4.
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**Proof.** First observe that \( x_{2P} = (x_P^2 - A)^2 / 4y_P^2 \), then a detailed elementary 2-adic analysis shows that if \( P \) is of odd order, then \( 2^4 \) divides \( A \).
Williams Curves

\[ E_m : y^2 = x^3 - (3m^4 + 24m)x + (-2m^6 + 40m^3 + 16) \]
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Remark. \( E_m \) is related to the existence of a pure cubic unit with rational summand \( x = m \). \((x + yD^{1/3} + zD^{2/3})/3\).
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**Remark** $P_m = (3m^2, 4(m^3 - 1))$ is of order 3 on $E_m$. 
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Remark \( P_m = (3m^2, 4(m^3 - 1)) \) is of order 3 on \( E_m \).

Theorem (Herrmann-W, 2003)

For all integers \( m \neq 1 \),

\[ T(E_m) \cong C_3. \]

Note: \( E_1 \) is singular
(Start of) Proof. Because $E_m$ has a point of order 3, Mazur’s theorem implies $T(E_m)$ is one of

$$C_3, C_6, C_9, C_{12}, C_2 \times C_6.$$
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$F = 0$ is a curve of genus 0, leading to

$$t(t^2 - 3m) = 2, \quad t \in \mathbb{Z}$$

and eventually to $m = 1$. 
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with

$$a_8 = -27Y^2$$

$$a_7 = 36Y^4 + 288Y$$

$$a_6 = 516Y^6 - 1248Y^3 - 1536$$

$$a_5 = 702Y^8 - 4320Y^5 + 13284Y^2$$

$$a_4 = -954Y^{10} - 11232Y^7 - 27648Y^4 + 9216Y$$

$$a_3 = -3372Y^{12} + 96Y^9 + 322560Y^6 - 270336Y^3 + 12288$$

$$a_2 = -3564Y^{14} + 49248Y^{11} - 622080Y^8 + 165888Y^5 + 331776Y^2$$

$$a_1 = -1719Y^{16} + 65376Y^{13} + 548352Y^{10} - 589824Y^7 + 626688Y^4 - 589824Y$$

$$a_0 = -323Y^{18} + 24672Y^{15} - 823296Y^{12} + 1586176Y^9 - 1265664Y^6 + 196608Y^3 + 262144.$$
Part II: The Rank of $E$
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The Mordell-Weil Theorem The group $E(\mathbb{Q})$ is finitely generated.

Proof

- properties of height functions on $E$
- $[E : 2E]$ is finite
- Descent theorem
Computing the Rank of $y^2 = x^3 + Ax$
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$$E(\mathbb{Q}) \cong \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}/p_1^{n_1} \mathbb{Z} \times \mathbb{Z}/p_k^{n_k} \mathbb{Z}$$

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If $G = \mathbb{Z}/p_i^{n_i} \mathbb{Z}$, then

$$[G : 2G] = \begin{cases} 2 & \text{if } p = 2, \\ 1 & \text{otherwise,} \end{cases}$$
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where $q$ is the number of $i$ with $p_i = 2$. 
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Need to understand $[2] : E \to E$. 
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Given $E : y^2 = x^3 + Ax$, define

$$E : y^2 = x^3 - 4Ax.$$  

Notice that $E$ is given by $y^2 = x^3 + 2^4Ax$, and

$\psi : E \to E$, given by

$$\psi(x, y) = (x/4, y/8),$$

is an isomorphism.
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Given $E : y^2 = x^3 + Ax$, define

$$\overline{E} : y^2 = x^3 - 4Ax.$$  

Notice that $\overline{E}$ is given by $y^2 = x^3 + 2^4Ax$, and $\psi : \overline{E} \to E$, given by

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**Lemma** For $P = (x, y) \in E$, define

$$\phi(P) = \begin{cases} 
O_E & \text{if } P = O, P = (0, 0) \\
(x + A/x, y/x(x - A/x)) & \text{otherwise.}
\end{cases}$$

Then $\phi$ is a homomorphism from $E$ to $\overline{E}$ with $Ker(\phi) = \{O, (0, 0)\}$.  

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Then $\phi$ is a homomorphism from $E$ to $\overline{E}$ with $Ker(\phi) = \{O, (0, 0)\}$.

$\phi : E \to \overline{E}$ is similarly defined.
Factoring [2]

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$$[2]P = \psi \phi \phi(P).$$

**Lemma**

$$2^{r+2} = [E(\mathbb{Q}) : \overline{\phi(E(\mathbb{Q}))}] \cdot [E(\mathbb{Q}) : \phi(E(\mathbb{Q}))].$$
One More Map
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For \( x \in \mathbb{Q}^* \), let \([x]\) denote the coset of \( x \) in \( \mathbb{Q}^*/\mathbb{Q}^*2 \).

For example \([9/8] = 1/2\).
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Define $\alpha : E(\mathbb{Q}) \to \mathbb{Q}^*/\mathbb{Q}^{*2}$ by

$$\alpha(O) = 1, \alpha((0,0)) = [A],$$

and for $P = (x,y)$ with $x \neq 0$,

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Lemma \( \alpha(E(\mathbb{Q})) \cong E(\mathbb{Q})/\overline{\phi(E(\mathbb{Q}))} \).
A Computational Tool for the Rank

\[ E = E_A : y^2 = x^3 + Ax \]
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Corollary \[ 2^{r+2} = |\alpha(E(\mathbb{Q}))| \cdot |\overline{\alpha}(\overline{E}(\mathbb{Q}))| \].
A Computational Tool for the Rank

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**Corollary** \[ 2^{r+2} = |\alpha(E(\mathbb{Q}))| \cdot |\overline{\alpha}(\overline{E}(\mathbb{Q}))|. \]

**Theorem** The group \( \alpha(E) \) consists of \( 1, [A], \pm [x] \) (if \(-A = x^2 \) for some \( x \in \mathbb{N} \)), and those \([d]\) such that \( d \) is a (positive or negative) divisor of \( A \) \( (d \neq 1, A) \) with the property that

\[ ds^4 + (A/d)t^4 = u^2 \]

is solvable in positive integers \( S, T, U \), with \( \gcd(A/d, S) = 1 \).

A similar statement holds for \( \overline{\alpha}(\overline{E}) \).
An Example:

\[ E : y^2 = x^3 - 17x \] and \[ \overline{E} : y^2 = x^3 + 68x. \]
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$1, -17 \in \alpha(E)$, and we need only check $-1, 17$:

$$-S^4 + 17T^4 = U^2, \quad 17S^4 - T^4 = U^2$$

are solvable in positive integers, so

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Therefore, \( 2^{r+2} = 4 \cdot 4 = 16 \), hence \( r = 2 \).
A Theorem of Blair Spearman
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**Theorem** If $p$ is a rational prime of the form $p = u^4 + v^4$, then the rank over $\mathbb{Q}$ of

$$E_p : y^2 = x^3 - px$$

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$$E_p : y^2 = x^3 - px$$

is equal to 2.

**Proof** Compute $|\alpha(E_p)|$ and $|\overline{\alpha(E_p)}|$.

We automatically have $1, -p \in \alpha(E_p)$, so we just need to show $-1, p \in \alpha(E_p)$, which means showing that

$$-S^4 + pT^4 = U^2$$

is solvable with $\gcd(S, p) = 1$, and that

$$pS^4 - T^4 = U^2$$

is solvable with $\gcd(S, -1) = 1$. 
Put \((S, T, U) = (u, 1, v^2)\) in the first case and 
\((S, T, U) = (1, u, v^2)\) in the second case.
It follows that \(|\alpha(E_p)| = 4|\).
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Put \((S, T, U) = (u - v, 1, 2(u^2 - uv + v^2))\).

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\[(p = u^4 + v^4 \Rightarrow \gcd(S, 2p) = (u - v, 2p) = 1)\]

Thus, \(|\overline{\alpha(E_p)}| = 4|, and \(2^{r+2} = 4 \cdot 4 = 16|, and

\[
\text{rank}_{E_p} = 2.
\]
III. Integer Points on Elliptic Curves

**Theorem (Siegel, 1929)** Let $F \in \mathbb{Z}[X,Y]$. If the curve $F(X,Y) = 0$ represents a curve of genus 1, then there are only finitely many integers $x, y$ for which $F(x, y) = 0$. 
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Theorem (Baker and Coates, 1970) Let $F \in \mathbb{Z}[X,Y]$ of total degree $n$ and height $H$. If the curve $F(X,Y) = 0$ represents a curve of genus 1, and $x, y$ are integers satisfying $F(x, y) = 0$, then

$$\max(|x|, |y|) < \exp \exp \exp((2H)^{10^{n^{10}}}).$$
Computing All Integer Points on a Curve
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- Packages exist which have programs to compute all integer points on an elliptic curve: MAGMA, PARI, KASH, SIMATH.
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1. Compute generators for

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1. Compute generators for

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2. S. David’s bound for linear forms in elliptic logarithms to get a (large) bound for \( M \):

\[ P = P_T + k_1 P_1 + \cdots + k_r P_r \]

and \( P \in E(\mathbb{Z}) \) implies \( k_i < M \).
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\[ u = \frac{375494528127162193105504069942092792346201}{62159877768644257535639389356838044100} \]
A Hybrid Theorem

**Theorem (W, 2010)** Let $N$ denote a square-free positive integer, and let

$$E : y^2 = x^3 - Nx.$$  

Then there are at most

$$48 \cdot 3^{\omega(N)}$$

integer points $(X,Y)$ on $E$ with

$$|X| > \max_{D|N,D>1} \frac{6|N/D|^{20} \epsilon_D^{23}}{D^6},$$

where $\omega(D)$ is the number of prime factors of $D$ and $\epsilon_D$ is the fundamental unit in $\mathbb{Q}(\sqrt{D})$. 
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Main Tool Siegel’s method for irrationality measure in Diophantine Approximation applied to algebraic numbers of degree 4.
Integral Points on Spearman’s Curves
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**Exercise** The maximum of 4 is attained!!
An Extension of Spearman’s Theorem

**Theorem (W, 2010)** Let \( p \) denote an odd prime, and let \( E_p : y^2 = x^3 - px \). Classify the integer points \((x, y)\) on \( E_p \) with \( y > 0 \) as follows:
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**iii.** If \( \epsilon_p = T + U\sqrt{p} \) satisfies \( \text{Norm}(\epsilon_p) = -1 \) and \( U = u^2 \) for some integer \( u \), then \((x, y) = (pu^2, puT) \in E_p\).
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If $E_p$ contains two integer points $(x, y)$ with $y > 0$, then the rank of $E_p$ is 2 except possibly if the two integer points are of type ii. and iii.
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**Example** Spearman’s curves have two points of type ii. If $p = 577$, $E_p$ has one point of each type and by the Theorem, $\text{rank}(E_{577}) = 2$. 

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Reduction to a Thue Equation

All integer solutions \((x, y)\) to
\[
x^2 - (2^{2m} + 1)y^2 = -2^{2m} \quad (\ast)
\]
arise from
\[
x + y\sqrt{2^{2m} + 1} = \pm (\pm 1 + \sqrt{2^{2m} + 1})(2^m + \sqrt{2^{2m} + 1})^{2i}
\]
for some \(i \geq 0\).

Put
\[
T_k + U_k\sqrt{2^{2m} + 1} = (2^m + \sqrt{2^{2m} + 1})^k
\]

A solution \((x, y)\) to \((\ast)\) with \(y = Y^2\)
is equivalent to
\[
Y^2 = T_{2k} \pm U_{2k} = (T_k \pm U_k)^2 + (2aU_k)^2.
\]
\[ Y^2 = (T_k \pm U_k)^2 + (2aU_k)^2, \]
hence there are coprime positive integers \( r, s \) such that
\[ Y = r^2 + s^2, \quad T_k \pm U_k = r^2 - s^2, \quad 2aU_k = 2rs, \]
with \( r \) even and \( s \) odd. Put \( R = r/a \).
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with \( r \) even and \( s \) odd. Put \( R = r/a \).

**Solve** for \( T_k, U_k \), substitute \((x, y) = (T_k, U_k)\) into \( x^2 - (2^{2m} + 1)y^2 = \pm 1 \):

**Thue equation:**
\[ Y^2 = (T_k \pm U_k)^2 + (2aU_k)^2, \]

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\textbf{Solve} for \( T_k, U_k \), substitute \( (x, y) = (T_k, U_k) \) into \( x^2 - (2^{2m} + 1)y^2 = \pm 1 : \)

\textbf{Thue equation:}

\[ s^4 - 2s^3R - 6a^2s^2R^2 + 2a^2sR^3 + a^4R^4 = \pm 1 \]

\((R = r/a \text{ and } a = 2^{m-1}).\)
Akhtari’s Theorem (to appear in Acta Arithmetica)

Let $F(x, y)$ be an irreducible binary quartic form with integer coefficients that splits in $\mathbb{R}$. If $J_F = 0$, then the inequality

$$|F(x, y)| = 1$$

has at most 12 positive integer solutions $(x, y)$. 
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**Proof** Siegel’s method (1929), elaborated by Evertse (1983).
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**Corollary**

For all $m \geq 0$, the equation

$$X^2 - (2^{2m} + 1)Y^4 = -2^{2m}$$

has at most 3 solutions in coprime positive integers $(X, Y) \neq (1, 1)$. 

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Yuan’s Theorem

Let $A > 0$, $B$ and $N$ be rational integers, and

$$F(X, Y) = BX^4 - AX^3Y - 6BX^2Y^2 + AXY^3 + BY^4.$$ 

If $A > 308B^4$, then all coprime integer solutions $(x, y)$ to the inequality

$$|F(x, y)| \leq N$$

satisfy

$$x^2 + y^2 \leq \max \left( \frac{25A^2}{64B^2}, \frac{4N^2}{A} \right).$$
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\[ x^2 + y^2 \leq \max \left( \frac{25A^2}{64B^2}, \frac{4N^2}{A} \right). \]

**Proof** The hypergeometric method is used to obtain an irrationality measure for a class of algebraic numbers, for approximations $p/q$ with $p, q$ in an imaginary quadratic field.
Observation 1

If \((X, Y) \neq (1, 1)\) is a solution in coprime positive integers to

\[ X^2 - (2^{2m} + 1)Y^4 = -2^{2m}, \]

with \(Y = r^2 + s^2, \ r > s > 0, \) and \(a = 2^{m-1}, \)

then

\[ \pm X \pm 2ai = (1 + 2ai)(s \pm ri)^4. \]
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$$\pm X \pm 2ai = (1 + 2ai)(s \pm ri)^4.$$

proof Recall

$$s^4 - 2s^3R - 6a^2s^2R^2 + 2a^2sR^3 + a^4R^4 = \pm 1.$$

Diagonalize this over the Gaussian integers:

$$(1 + 2ai)(s + ri)^4 - (1 - 2ai)(s - ri)^4 = \pm 4ai.$$
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Put \(X_0 = (1 + 2ai)(s + ri)^4 + (1 - 2ai)(s - ri)^4\), the result follows from \(X_0 = X\).
Observation 2 (The Gap Principle)

If \((X_1, Y_1), (X_2, Y_2)\) are two coprime positive integer solutions to

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**proof** For \(j = 1, 2\) and \(Y_j = s_j^2 + r_j^2\), we have

\[ (1 + 2ai)(s_j + r_ji)^4 - (1 - 2ai)(s_j - r_ji)^4 = \pm 4ai. \]
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Let \(\omega = \frac{1-2ai}{1+2ai}\), use the fact that
\[
| \omega - \left( \frac{s_j + r_ji}{s_j - r_ji} \right)^4 | = \frac{4a}{\sqrt{1 + 4a^2Y_j^2}}
\]
is very small for both \(j = 1, 2\).
The Main Argument

Suppose that \((X_1, Y_1), (X_2, Y_2), (X_3, Y_3)\) are co-prime positive integer solutions to

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with \(Y_3 > Y_2 > Y_1 > 1\), \(Y_j = s_j^2 + r_j^2\) \((j = 1, 2, 3)\).
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\[ X_1 \pm 2ai = (1 \pm 2ai)(s_1 \pm r_1i)^4, \]

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The Main Argument

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\[ X_1 \pm 2ai = (1 \pm 2ai)(s_1 \pm r_1 i)^4, \]

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\[ (1+2ai)(s_3+r_3 i)^4 - (1-2ai)(s_3-r_3 i)^4 = \pm 4ai. \]

Using the above, the following is easy to show:
\[ \gamma - \bar{\gamma} = \pm 4Y_1^4 ai, \]

with

\[ \gamma = (X_1 \pm 2ai)(s_1 - r_1 i)^4 (s_3 + r_3 i)^4. \]
\[ \gamma - \bar{\gamma} = \pm 4Y_1^4 ai, \]

with

\[ \gamma = (X_1 \pm 2ai)(s_1 - r_1i)^4(s_3 + r_3i)^4. \]

Define \((x, y)\) by

\[ x + yi = (s_1 - r_1i)(s_3 + r_3i), \]

then

\[ | (X_1 \pm 2ai)(x+yi)^4 - (X_1 \mp 2ai)(x-yi)^4 | = 4aY_1^4, \]
\[ \gamma - \overline{\gamma} = \pm 4Y_1^4 ai, \]

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\[ |(X_1 \pm 2ai)(x+yi)^4 - (X_1 \mp 2ai)(x-yi)^4| = 4aY_1^4, \]

i.e.

\[ |\mp ax^4 - 2X_1x^3y \pm 6ax^2y^2 + 2X_1xy^3 \mp ay^4| = aY_1^4. \]
\[ \gamma - \overline{\gamma} = \pm 4Y_1^4 ai, \]

with

\[ \gamma = (X_1 \pm 2ai)(s_1 - r_1i)^4(s_3 + r_3i)^4. \]

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\[ |\mp ax^4 - 2X_1x^3y \pm 6ax^2y^2 + 2X_1xy^3 \mp ay^4| = aY_1^4. \]

This is a Thue equation of the form in Yuan's theorem with

\[ B = \pm a, A = 2X_1, N = aY_1^4. \]
The hypothesis in Yuan’s theorem:

\[ A > 308B^4 \]
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Recall

\[ Y_1^2 = T_{2k} \pm U_{2k}. \]

Similarly

\[ X_1 = (1 + 4a^2)U_{2k} \pm T_{2k}. \]
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\[ X_1 = (1 + 4a^2)U_{2k} \pm T_{2k}. \]

Assume \( k > 1 \) (regard \( k = 1 \) as an exercise).

Then

\[ A = 2X_1 \geq 2(4a^2 + 1)U_4 - 2T_4 = 16a(4a^2 + 1)(8a^2 + 1) - 4(8a^2 + 1)^2 > 308a^4 = 308B^4. \]
The conclusion of Yuan’s theorem gives

\[ x^2 + y^2 \leq \max \left( \frac{100X_1^2}{64a^2}, \frac{4a^2Y_1^8}{2X_1} \right), \]
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\[ x^2 + y^2 = (r_1^2 + s_1^2)(r_3^2 + s_3^2) = Y_1Y_3 \geq 16Y_1^{10}. \]
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The inequality \( X_1^2 < (4a^2 + 1)Y_1^4 \) is used to derive a contradiction from these two inequalities.
Theorem For all $m \geq 0$, there are at most 2 solutions in coprime positive integers $(X,Y) \neq (1,1)$ to the equation

$$X^2 - (2^{2m} + 1)Y^4 = -2^{2m}.$$
**Theorem** For all \( m \geq 0 \), there are at most 2 solutions in coprime positive integers \((X, Y) \neq (1, 1)\) to the equation

\[
X^2 - (2^{2m} + 1)Y^4 = -2^{2m}.
\]

**Conjecture** For all \( m \geq 3 \), there are NO solutions in coprime positive integers \((X, Y)\) to the equation

\[
X^2 - (2^{2m} + 1)Y^4 = -2^{2m}
\]

other than \((X, Y) = (1, 1)\).