Electrical Impedance Tomography: 3D reconstructions using scattering transforms

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Potential equation and Calderón problem

- 3D domain $\Omega$, boundary $\partial \Omega$, outward normal $\nu$
- conductivity $\sigma$ (real valued)

\[ f \cdot \nu = \sigma \partial \nu u \quad \text{on} \quad \partial \Omega \]
\[ \Rightarrow \quad \text{potential} \quad u : \nabla \cdot \sigma \nabla u = 0 \quad \text{in} \quad \Omega, \]
\[ u = f \quad \text{on} \quad \partial \Omega. \]
\[ \Rightarrow \quad \text{current} \quad g = \sigma \partial \nu u \quad \text{on} \quad \partial \Omega \]

Dirichlet-to-Neuman map (voltage-to-current map) $\Lambda_\sigma : f \mapsto g$.

**Calderón Problem**: Is $\sigma$ uniquely determined by $\Lambda_\sigma$ and does there exist an algorithm to compute $\sigma$ from $\Lambda_\sigma$?
Potential equation and Calderón problem

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- conductivity $\sigma$ (real valued)

$$\begin{align*}
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
u &= f \text{ on } \partial \Omega. \\
\Rightarrow \text{ current } g = \sigma \partial_\nu u \text{ on } \partial \Omega \\
\end{align*}$$

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Voltage $f$ on $\partial \Omega$\
$\Rightarrow$ potential $u$:
$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \quad \Omega,$$
$$u = f \quad \text{on} \quad \partial \Omega.$$

$\Rightarrow$ current $g = \sigma \partial_\nu u$ on $\partial \Omega$

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Ca\-derón Problem: Is $\sigma$ uniquely determined by $\Lambda_\sigma$ and does there exist an algorithm to compute $\sigma$ from $\Lambda_\sigma$?
Short and incomplete history


- **2D**
  1996 Nachman: Uniqueness and reconstruction for $W^{2,p}(\Omega)$ conductivities.
  1997 Brown-Torres: Uniqueness for $W^{1,p}(\Omega)$ conductivities.
  2001 Barceló-Barceló-Ruiz: Stability for $C^{1+\epsilon}$ conductivities.
  2001 Knudsen-Tamasan: Reconstruction for $C^{1+\epsilon}$ conductivities.
  2005 Astala-Päivärinta: Uniqueness and reconstruction for $L^\infty(\Omega)$ conductivities.

- **3D**
  1987-88 Novikov, Nachman-Sylvester-Uhlmann, Nachman: Uniqueness for conductivities with 2 derivatives and explicit high frequency reconstruction algorithm. Multidimensional $\overline{\partial}$-bar equation.
  2006 Cornean-Knudsen-Siltanen: Low frequency reconstruction algorithm.
  2011 Delbary-Hansen-Knudsen: Implementation of more accurate numerical reconstruction
Outline

- Reconstruction algorithm
- Simplifications
- Implementation
- Numerical results
- Conclusion and outlook
Assumptions

- $\Omega = B(0, 1)$, unit ball in $\mathbb{R}^3$ (to simplify implementation).

- $\sigma \in C^\infty(\bar{\Omega})$ (can be less regular).

- $\sigma = 1$ in the neighborhood of $\partial\Omega$ (to simplify implementation).

$\Rightarrow \sigma$ extended by 1 in $\mathbb{R}^3 \setminus \bar{\Omega}$
For \( f \in H^{1/2}(\partial \Omega) \), if \( u \) is the solution to the potential equation

\[
\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \quad \Omega,
\]

\[
u = f \quad \text{on} \quad \partial \Omega,
\]

then \( v = \sigma^{1/2}u \) is the solution to the Schrödinger equation

\[
-\Delta v + qv = 0 \quad \text{in} \quad \Omega,
\]

\[
v = f \quad \text{on} \quad \partial \Omega,
\]

where \( q = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}} \).
Reconstruction Algorithm

The Schrödinger equation

For $f \in H^{1/2}(\partial \Omega)$, if $u$ is the solution to the potential equation

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \quad \Omega,$$

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Reconstruction Algorithm

Complex Geometrical Optics solutions (CGO)

CGO solutions $\psi_\zeta$, $\zeta \in \mathbb{C}^3$, $\zeta \cdot \zeta = 0$

$$(-\Delta + q)\psi_\zeta = 0 \text{ in } \mathbb{R}^3,$$

$$\psi_\zeta(x) \approx e^{ix \cdot \zeta} \text{ for large } |x| \text{ or } |\zeta|.$$

Small or large $\zeta \Rightarrow$ existence and uniqueness of the solutions.
$\xi \in \mathbb{R}^3$, $\zeta \in \mathbb{C}^3$, $\zeta^2 = (\xi + \zeta)^2 = 0$

$$t(\xi, \zeta) = \int_{\mathbb{R}^3} q(x)e^{-ix \cdot (\xi + \zeta)} \psi_\zeta(x) \, dx.$$ 

$$\psi_\zeta(x) \approx e^{ix \cdot \zeta} \text{ for large } \zeta \Rightarrow |\hat{q}(\xi) - t(\xi, \zeta)| = O \left( \frac{1}{|\zeta|} \right)$$

How to compute $t$ from $\Lambda_\sigma$?
Reconstruction Algorithm
Scattering transform

\[ \xi \in \mathbb{R}^3, \ \zeta \in \mathbb{C}^3, \ \zeta^2 = (\xi + \zeta)^2 = 0 \]

\[ t(\xi, \zeta) = \int_{\mathbb{R}^3} q(x)e^{-ix \cdot (\xi + \zeta)} \psi_{\zeta}(x) \, dx. \]

\[ \psi_{\zeta}(x) \approx e^{ix \cdot \zeta} \text{ for large } \zeta \Rightarrow |\hat{q}(\xi) - t(\xi, \zeta)| = \mathcal{O}\left(\frac{1}{|\zeta|}\right) \]

How to compute \( t \) from \( \Lambda_\sigma \)?
Reconstruction Algorithm
Scattering transform

\( \xi \in \mathbb{R}^3, \zeta \in \mathbb{C}^3, \zeta^2 = (\xi + \zeta)^2 = 0 \)

\[
t(\xi, \zeta) = \int_{\mathbb{R}^3} q(x)e^{-i x \cdot (\xi + \zeta)} \psi_\zeta(x) \, dx.
\]

\( \psi_\zeta(x) \approx e^{i x \cdot \zeta} \) for large \( \zeta \) \Rightarrow \( |\hat{q}(\xi) - t(\xi, \zeta)| = \mathcal{O}\left(\frac{1}{|\zeta|}\right) \)

How to compute \( t \) from \( \Lambda_\sigma \)?
Reconstruction Algorithm

Scattering transform

\[ \zeta^2 = (\xi + \zeta)^2 = 0 \]

\[ t(\xi, \zeta) = \int_{\mathbb{R}^3} q(x)e^{-ix \cdot (\xi + \zeta)} \psi_\zeta(x) \, dx. \]

Green's formula in \( \Omega \)

\[ t(\xi, \zeta) = \int_{\partial \Omega} e^{-ix \cdot (\xi + \zeta)} [(\Lambda_\sigma - \Lambda_1) \psi_\zeta](x) \, ds(x). \]
\[ \zeta^2 = (\xi + \zeta)^2 = 0 \]

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\[ t(\xi, \zeta) = \int_{\partial \Omega} e^{-ix \cdot (\xi + \zeta)} [(\Lambda_{\sigma} - \Lambda_1) \psi_{\zeta}](x) \, ds(x). \]
Faddeev Green’s function

\[ G_\zeta(x) = \frac{e^{ix \cdot \zeta}}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix \cdot \xi}}{|\xi|^2 + 2\xi \cdot \zeta} \, d\xi, \quad x \in \mathbb{R}^3 \setminus \{0\}, \]

\( G_\zeta \) fundamental solution to the Laplace equation

\[ -\Delta G_\zeta = \delta_0 \quad \text{with} \quad G_\zeta(x) \sim e^{ix \cdot \zeta} \quad \text{for large} \quad |x|. \]
Reconstruction Algorithm

Boundary integral equation for the CGO

\[ (-\Delta + q)\psi_\zeta = 0 \text{ in } \mathbb{R}^3, \]

\[ \psi_\zeta(x) \approx e^{ix \cdot \zeta} \text{ for large } |x| \text{ or } |\zeta|. \]

Green’s formula in \( \mathbb{R}^3 \setminus \overline{\Omega} \) and condition at infinity

\[ \psi_\zeta(x) + \int_{\partial \Omega} G_\zeta(x-y)[(\Lambda_\sigma - \Lambda_1)\psi_\zeta](y) \, ds(y) = e^{ix \cdot \zeta}, \quad x \in \partial \Omega, \]

i.e.

\[ \psi_\zeta(x) + [S_\zeta(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x) = e^{ix \cdot \zeta}, \quad x \in \partial \Omega, \]

with \( S_\zeta \) single layer operator.
Reconstruction Algorithm

Boundary integral equation for the CGO

\[ (-\Delta + q)\psi_{\zeta} = 0 \quad \text{in} \quad \mathbb{R}^3, \]
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Reconstruction Algorithm

$\Lambda_\sigma \overset{1}{\rightarrow} t(\xi, \zeta) \overset{2}{\rightarrow} q(x) \overset{3}{\rightarrow} \sigma(x)$

1. Solve

$$\psi_\zeta(x) + [S_\zeta(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x) = e^{ix \cdot \zeta}, \ x \in \partial \Omega$$

and compute

$$t(\xi, \zeta) = \int_{\partial \Omega} e^{-ix \cdot (\xi + \zeta)} [(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x) \, ds(x).$$

2. Compute $q$ by Inverse Fourier transform and the limit

$$\lim_{|\zeta| \to \infty} t(\xi, \zeta) = \hat{q}(\xi).$$

3. Solve

$$-\Delta \sigma^{1/2} + q\sigma^{1/2} = 0 \text{ in } \Omega,$$

$$\sigma^{1/2} = 1 \text{ on } \partial \Omega.$$
Reconstruction Algorithm

\[ \Lambda_\sigma \xrightarrow{1} t(\xi, \zeta) \xrightarrow{2} q(x) \xrightarrow{3} \sigma(x) \]

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Linearization step by step

• **Step 1:** $\Lambda_{\sigma} \mapsto t(\xi, \zeta)$ linearized around $\Lambda_{\sigma} = \Lambda_1 \Rightarrow$

$$t(\xi, \zeta) \simeq t^{\exp}(\xi, \zeta) = \int_{\partial \Omega} e^{-ix \cdot (\xi + \zeta)} [(\Lambda_{\sigma} - \Lambda_1)e^{iy \cdot \zeta}] (x) \, ds(x).$$

• **Step 2:** $t^{\exp}(\xi, \zeta) \simeq \hat{q}(\xi)$.

$\hat{q} \mapsto q$ linear.

• **Step 3:** $q \mapsto \sigma^{1/2} \mapsto \sigma$ linearized around $\sigma = 1 \Rightarrow$ Calderón’s formula

$$\sigma^{\text{app}}(x) = 1 - \frac{2}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{t^{\exp}(\xi, \zeta)}{|\xi|^2} e^{ix \cdot \xi} \, d\xi.$$
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  \]

- **Step 2**: \( t^{\exp}(\xi, \zeta) \simeq \hat{q}(\xi) \).
  \( \hat{q} \mapsto q \) linear.

- **Step 3**: \( q \mapsto \sigma^{1/2} \mapsto \sigma \) linearized around \( \sigma = 1 \) \( \Rightarrow \) Calderón’s formula
  \[
  \sigma^{\text{app}}(x) = 1 - \frac{2}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{t^{\exp}(\xi, \zeta)}{|\xi|^2} e^{ix \cdot \xi} \, d\xi.
  \]
Compute $t_{\text{exp}}$ and use it in Step 2 instead of $t$.

$$\Lambda_{\sigma} \xrightarrow{1} t_{\text{exp}}(\xi, \zeta) \xrightarrow{2} q_{\text{exp}}(x) \xrightarrow{3} \sigma_{\text{exp}}(x)$$
\( t^0 \) approximation

- Use the usual Green’s function \( G_0(x) = \frac{1}{4\pi|x|} \) instead of the Faddeev Green’s function and solve

\[
\psi^0_\zeta(x) + [S_0(\Lambda_\sigma - \Lambda_1)\psi^0_\zeta](x) = e^{ix \cdot \zeta}, \quad x \in \partial \Omega,
\]

with \( S_0 \) usual single layer operator.

- Use \( \psi^0_\zeta \) to compute

\[
t^0(\xi, \zeta) = \int_{\partial \Omega} e^{-ix \cdot (\xi + \zeta)}[\Lambda_\sigma - \Lambda_1)\psi^0_\zeta](x) \, ds(x).
\]

- Use \( t^0 \) in Step 2 instead of \( t \).

\[
\Lambda_\sigma \xrightarrow{1} t^0(\xi, \zeta) \xrightarrow{2} q^0(x) \xrightarrow{3} \sigma^0(x)
\]
Implementation

- $N$: positive integer
- $t_m$: increasing $N + 1$ zeros of the Legendre polynomial $P_{N+1}$
- $\theta_m = \arccos t_m$
- $\varphi_n = \pi n / (N + 1)$
- $2(N + 1)^2$ grid points on the unit sphere
  - $m = 0 \ldots N$, $n = 0 \ldots 2N + 1$
  - $x_{m,n} = (\sin \theta_m \cos \varphi_n, \sin \theta_m \sin \varphi_n, \cos \theta_m)$.

**Figure:** $N = 15 \rightarrow 512$ points
Implementation

- $\alpha_m = \frac{2(1 - t_m^2)}{(N + 1)^2 [P_N(t_m)]^2}$: weights of the Gauß-Legendre quadrature rule of order $N+1$ on $[-1, 1]$.

$\Rightarrow$ quadrature rule on the sphere (exact for spherical harmonics of degree less than or equal to $2N + 1$)

\[
\int_{\partial\Omega} \phi \, ds \simeq \frac{\pi}{N + 1} \sum_{m=0}^{N} \sum_{n=0}^{2N+1} \alpha_m \phi(x_{m,n}) , \phi \in C^0(\partial\Omega).
\]
Having $\Lambda_\sigma - \Lambda_1$, numerically compute

$$t^{\text{exp}}(\xi, \zeta) = \int_{\partial \Omega} e^{-i x \cdot (\xi + \zeta)} [(\Lambda_\sigma - \Lambda_1)e^{iy \cdot \zeta}](x) \, ds(x)$$

with previous quadrature rule.
Implementation
Computing $t^0$

- Boundary integral equation solved using a Nyström-like method: based on a quadrature rule to compute $S_0 \phi$ for $\phi \in C^0(\partial \Omega)$

- $Y^m_n$ are the eigenvectors of $S_0$

\[
\frac{1}{4\pi} \int_{\partial \Omega} \frac{Y^m_n(y)}{|x-y|} \, ds(y) = \frac{Y^m_n(x)}{2n+1}, \quad x \in \partial \Omega.
\]
Implementation
Computing $t^0$

- $L^2(\partial \Omega)$ orthogonal projection operator on the span of spherical harmonics of degree less than or equal to $N$

\[ T_N \phi = \sum_{n=0}^{N} \sum_{m=-n}^{n} \langle \phi, Y_{n}^{m} \rangle Y_{n}^{m}, \quad \phi \in L^2(\partial \Omega) \]

- Inner product approximated by quadrature rule

$\Rightarrow$ hyperinterpolation operator

\[ L_N \phi = \frac{\pi}{N + 1} \sum_{n=0}^{N} \sum_{m=-n}^{n} \sum_{k=0}^{N} \sum_{\ell=0}^{2N+1} \alpha_{k,\ell} \phi(x_{k,\ell}) Y_{n}^{-m}(x_{k,\ell}) Y_{n}^{m}, \quad \phi \in C^0(\partial \Omega). \]
⇒ \( S_0 \phi \) approximated by \( S_0 L_N \phi \)

\[
[S_0 \phi](x) \simeq \frac{1}{4(N + 1)} \sum_{n=0}^{N} \sum_{k=0}^{N} \sum_{\ell=0}^{2N+1} \alpha_k \phi(x_k \ell) P_n(x_k \ell \cdot x) , \ x \in \partial \Omega.
\]

- Approximate the solution to

\[
\psi_0^\zeta(x) + [S_0(\Lambda_\sigma - \Lambda_1)\psi_0^\zeta](x) = e^{ix \cdot \zeta} , \ x \in \partial \Omega,
\]

by the solution to

\[
[I + S_0 L_N(\Lambda_\sigma - \Lambda_1)L_N]\psi^N(x) = e^{ix \cdot \zeta} , \ x \in \partial \Omega.
\]

(sums up to a finite dimensional linear system)
Implementation

Computing $t^0$

- Convergence rates: for any $s > 5/2$

$$\|\psi^N - \psi_0^\zeta\|_{H^s(\partial\Omega)} \leq \frac{C}{N^{s-5/2}} \|e^{ix\cdot\zeta}\|_{H^s(\partial\Omega)},$$

where $C$ depends only on $s$.

- Having computed $\psi_0^\zeta$, compute $t^0$ using the quadrature rule.
From the theory: $\lim_{|\zeta| \to \infty} t(\xi, \zeta) = \hat{q}(\xi)$.

Not true anymore for $t^{\text{exp}}$ or $t^0$: divergence when $|\zeta| \to \infty$.

Moreover: $e^{ix \cdot \zeta} \Rightarrow$ exponentially growing terms $\Rightarrow$ numerical instabilities.

$\Rightarrow \zeta$ chosen of minimal norm.
Inverse Fourier Transform

- Computed using a FFT.

- \( \hat{q}(\xi) \) computed on an equidistant mesh in a box \([-\xi_{\text{max}}, \xi_{\text{max}}]^3\)

\[ \Rightarrow q(x) \text{ on an equidistant } x \text{ grid in } [−1, 1]^3. \]

- Number \( N^3 \) of points in grid must satisfy

\[ \xi_{\text{max}} = N \frac{\pi}{2}. \]

\[ \Rightarrow \text{Upper limit for the resolution in the } x\text{-mesh in terms of the mesh-size } h = \frac{\pi}{\xi_{\text{max}}}. \]
Solving the Schrödinger equation

- \( q(x) \) interpolated on a tetrahedron mesh of the unit ball.

- Schrödinger equation solved using a FEM code of order 1.
Numerical examples
Radially symmetric conductivity

Figure: Approximations of $\hat{q}$.

Figure: Reconstructions of $\sigma$ with truncation: left $\xi_{\text{max}} = 8$, right $\xi_{\text{max}} = 9$. 
Numerical examples
Radially symmetric conductivity: noisy data

Figure: Approximations of $\hat{q}$ in case of noise in the data: left 0.1% noise, middle 1% noise, and right 5% noise.
Figure: Reconstructions of $\sigma$ with truncation: left 0.1% noise and $\xi_{\text{max}} = 8$, middle 1% noise and $\xi_{\text{max}} = 7$, and right 5% noise and $\xi_{\text{max}} = 6$.

Truncation of the scattering transform $\Rightarrow$ non-linear regularization strategy.
Numerical examples
Non radially symmetric conductivity

Figure: Left: 3D plot of phantom. Middle: profile of the conductivity $\sigma$ in the $(Oxy)$ plane. Right: support of $\sigma^{-1}$. 
Numerical examples
Non radially symmetric conductivity

Figure: Upper row $\xi_{\text{max}} = 6$, lower row $\xi_{\text{max}} = 8$. Left $\sigma^0$, middle $\sigma^{\text{exp}}$, and right $\sigma^{\text{app}}$. 
Numerical examples
Non radially symmetric conductivity: noisy data

Figure: Reconstructions with different noise levels: left column $0\%$, middle column $0.1\%$ and right column $1\%$. Upper row $\sigma^{\text{app}}$, middle row $\sigma^{\text{exp}}$, and lower row $\sigma^0$. Truncation is at $\xi_{\text{max}} = 6$. 
Conclusion and outlook

- Three different numerical simplifications and implementations.
- Fast reconstructions (\(\sim 1\) min).
- Contrast not reliable.
- \(t^0\) does not give better reconstructions.
- Implementation for \(t^0\) ⇒ implementation for \(t\) easily follows.
- Implementation can be adapted to more general domains \(\Omega\).

- Study more complex 3D numerical examples.
- \(t\) implementation.
- More general domains \(\Omega\).
- Understanding spectral cut-off as regularization strategy.
- Limited data aperture.
- Real data.

Thank you
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F. Delbary (DTU)
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