Compressive MUSIC for Diffuse Optical Tomography using Joint Sparsity

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Contents

- Motivation
- Compressed sensing with joint sparsity
- Compressive MUSIC (CS-MUSIC)
- Applications to DOT
- Summary
Compressed sensing

- Incoherent projection
- Underdetermined system
- Sparse unknown vector

\[ M \times 1 \text{ measurements} \quad = \quad A \quad \text{signal} \quad \times \quad x \]

\[ M \approx K \log(N) \ll N \]

Figure from Dr. Dror Baron
Compressed Sensing Dynamic MRI

Cardiac MR

fMRI

k-t FOCUSS for dynamic CS-MRI
(Jung et al, PMB, 2007, Jung et al, MRM, 2009)
Radial k-t FOCUSS
(Jung et MRM, 2010)

iRadon 16 views

Radial k-t FOCUSS with ME/MC from 16 views
k-t FOCUSS for Cardiac T2 Mapping (Feng et al, 2011)

6 x accel. Conventional method 6 x accel. k-t FOCUSS
**k-t FOCUSS** for Cardiac T2 Mapping

(Feng et al, 2011)

1.8 x accel.  **GRAPPA**  6 x accel.  **k-t FOCUSS**
CS + Parallel Imaging

Real Brain

Coil sensitivity

k-data

Recon.

IFFT
CS + Parallel Imaging

How to solve aliasing?

Real Brain

Coil sensitivity

k-data

MR scan

IFFT
Compressed Sensing for Joint Sparse Signals

**Multiple measurement vector (MMV) problem**

\[
B \quad m = m \quad A \quad X
\]

minimize \[ \|X\|_0 \]

subject to \[ B = AX. \]

\[ \|X\|_0 \] denote the number of nonzero rows

\[ r : \text{number of snapshots} \]

\[ m: \text{number of sensor elements} \]
MMV for Medical Imaging

• Parallel MRI + CS

• EEG/MEG

• Diffuse optical tomography

• Wave inverse scattering
Applications of DOT

Main applications: Molecular imaging, Neuroimaging

A. Custo et al. NeuroImage. 2010

Hitachi NIRS System


R. Weissleder et al. Radiology. 2001

A. Yodh Group
Diffusion Equation

The movement of photon can be described by

\[ \frac{1}{c} \frac{\partial u(x, t)}{\partial t} = \nabla \cdot D(x) \nabla u(x, t) - \mu_a(x) u(x, t) + f(x, t) \]

mean path length of ‘s’ : \( \frac{1}{\mu_s} \)

a photon is absorbed after moving : \( \frac{1}{\mu_a} \)
Joint Sparse Model in DOT

\[ v(x; L) = \int g_0(x, x') u(x'; L) \Delta \mu_a(x') dx' \]
Joint Sparse Recovery Model for DOT
(Ye et al, IEEE TMI, 2011)

\[
\begin{align*}
\text{minimize} & \quad \|X\|_0 \\
\text{subject to} & \quad Y = AX
\end{align*}
\]

- \(r\) : number of snapshots
- \(m\) : number of sensor elements

\(\|X\|_0\) denotes the \# of nonzero rows
Exact & Non-iterative Reconstruction  
(Lee, Ye, 2008, Ye, Bresler, Lee, 2008)

- 1st step: estimate the active index set \( \Lambda \)

\[ \Lambda = \{ j \in \{1, 2, \ldots, n\} : \Delta \mu_a(x_j) \neq 0 \} \]

- 2nd step: reconstruct the \( \Delta \mu_a(x) \)

\[ \tilde{X} = A_{\Lambda}^\dagger Y \quad \rightarrow \quad \Delta \tilde{\mu}_a(x_{(j)}) = \frac{\sum_{L=1}^{r} \left( \tilde{u}(x_{(j)}; L) \right)^* \tilde{X}(j, L)}{\sum_{L=1}^{r} |\tilde{u}(x_{(j)}; L)|^2} \]

Foldy-Lax equation

\[ u(x_{(j)}; l) = u_0(x_{(j)}; l) - \sum_{i \neq j} g_0(x_{(j)}, x_{(i)}) u(x_{(i)}; l) \Delta \mu_a(x_{(i)}) \]

\[ i, j = 1, \ldots, k, \quad l = 1, 2, \ldots, r^o \]
Uniqueness Result of MMV

Definition
Given a matrix $A$, let $\text{spark}(A)$ denote the smallest number of linearly dependent columns of $A$.

Theorem (Chen, Huo (2006))

If a matrix $X$ satisfies $AX = B$ and

$$\|X\|_0 < \frac{\text{spark}(A) + \text{rank}(B) - 1}{2},$$

then $X$ is the unique solution to the problem [P0].

With increasing number of snapshots, more non-zero elements can be recovered.
Conventional MMV Algorithms

- **Compressive sensing approaches**
  - $p$-thresholding
  - S-OMP
  - *Convex relaxation with mixed norm*
  - ReMBo (Reduce Mmv and Boost)
  - *Model based CS using block sparsity*
  - M-FOCUSS
  - M-SBL
  - etc

- **Array signal processing approaches**
  - MUSIC
  - ESPRIT
  - IQML
  - Maximum likelihood
  - etc

*Probabilistic guarantee*
*Deterministic guarantee*
Counter Example

\[ \mu_s' = 7.5 \text{ cm}^{-1} \]
\[ \mu_a = 0.05 \text{ cm}^{-1} \]
\[ \Delta \mu_a = 0.05 \text{ cm}^{-1} \]

\[ SNR = 40 \text{ dB} \]
\[ \lambda = 7 \text{ mm} \]
\[ \lambda = 5 \text{ mm} \]
Why new MMV algorithm is necessary?

- **S-OMP, Convex relaxation using mixed norm**
  - **Worst case analysis**
    - no improvement over SMV

\[
\max_{i \in \text{supp} X} \| A_S^\dagger a_j \|_1 < 1
\]

\[
\| X \|_0 < \frac{1}{2} \left( \frac{1}{\mu} + 1 \right)
\]

- **Average case analysis**
  - improvement with increasing number of snapshot
  - Simulation results show saturation effects
Why new MMV algorithm is necessary?

- **ReMBo (Reduce MMV and Boost)**

---

**Theorem (Mishali, Eldar (2008))**

Let $\bar{X}$ be the unique $k$-sparse solution matrix of $AX = B$ with $k < \text{spark}(A)/2$. In addition, let $a \in \mathbb{R}^r$ be a random vector with an absolutely continuous distribution and define $b = Ba$ and $\bar{x} = \bar{X}a$. Then for a random SMV system $Ax = b$, we have

- For every $a$, the vector $\bar{x}$ is the unique $k$-sparse solution.
- $\mathbb{P}(\text{supp}(\bar{x}) = \text{supp}(\bar{X})) = 1$.

The performance of ReMBo is dependent on randomly chosen input vectors so that it takes long time to reproduce the exact solution and its $L0$-performance is same as the SMV problem.
Why new MMV algorithm is necessary?

- **MUSIC Algorithm**
  - If $\text{rank}(B) = k$, the following MUSIC criterion holds

\[
Q^* a_j = 0 \\
j \in \text{supp} X
\]

- **Dichotomy:**
  - Achieves $l_0$ bound when $\text{rank}(B) = k$

\[
\|X\|_0 < \text{spark}(A) - 1
\]

  - Fails when $\text{rank}(B) < k$
  - $\Rightarrow$ **coherent source problem**
Research Goal

- **The Best of Both Worlds**
  - At $\text{rank}(B)=k$, it should be reduced to MUSIC
  - At $\text{rank}(B) \to 1$, it should be reduced to CS
  - At all $\text{rank}(B)$, it should be superior to all existing methods
Theorem

Assume that $A \in \mathbb{R}^{m \times n}$ satisfies $0 < \delta_{2k-r+1}(A) < 1$. If $I_{k-r} \subset \text{supp} X$ and $A_{I_{k-r}}$ is a matrix which consists of columns whose indexes are in $I_{k-r}$. Then for any $j \in \{1, \cdots, N\} \setminus I_{k-r}$,

$$\text{rank}(Q^*[A_{I_{k-r}}, a_j]) = k - r$$

if and only if

$$j \in \text{supp} X$$

where $\text{supp} X = \{i : X^i \neq 0\}$.

For the canonical form MMV, $A \in \mathbb{R}^{m \times n}$ satisfies RIP with $0 \leq \delta_{2k-r+1} < 1$ if and only if

$$k < \frac{\text{spark}(A) + \text{rank}(B) - 1}{2}.$$ 


Corollary

Assume that $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{m \times r}$, $I_{k-r} \subseteq \text{supp}X$. Then,

$$a_j^* \left[ P_{R(Q)} - P_{R(P_{R(Q)}A_{l_{k-r}})} \right] a_j = 0$$

if and only if $j \in \text{supp}X$. 

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Diagram:
- $R(P_{R(Q)} - P_{R(Q^*A_{l_{k-r}})})$: (noise subspace for generalized MUSIC)
- $R(QQ^*A_{l_{k-r}})$: (noise subspace for MUSIC)
- $R(B)$: (signal subspace)
Theorem

Let \( U \in \mathbb{R}^{m \times r} \) and \( Q \in \mathbb{R}^{m \times (m-r)} \) consist of orthonormal columns such that \( R(U) = R(B) \) and \( R(Q)^\perp = R(B) \). Then the following properties hold:

(a) \( UU^* + P_{R(QQ^*A_{I_{k-r}})} \) is equal to the orthogonal projection onto \( R(B) + R(QQ^*A_{I_{k-r}}) \).

(b) \( QQ^* - P_{R(QQ^*A_{I_{k-r}})} \) is equal to the orthogonal projection onto \( R(Q) \cap R(QQ^*A_{I_{k-r}})^\perp \).

(c) \( QQ^* - P_{R(QQ^*A_{I_{k-r}})} \) is equal to the orthogonal complement of \( R([U \ A_{I_{k-r}}]) \) or \( R([B \ A_{I_{k-r}}]) \).
Compressive MUSIC Algorithm

1st Step: Compressive sensing step

Find $l_{k-r}$ with existing MMV-algorithm such as 2-thresholding and SOMP and let $S = l_{k-r}$.

Probabilistic performance guarantee

2nd Step: generalized MUSIC step

For $j \in \{1, \cdots, N\} \setminus l_{k-r}$, calculate $g_j^* P_{G_{l_{k-r}}}^\perp g_j = 0$:

- If $g_j^* P_{G_{l_{k-r}}}^\perp g_j = 0$, then add $j$ into $S$.

Deterministic performance guarantee
Compressive MUSIC (r=1)

Y = A

CS

MUSIC

Array Signal Processing
(deterministic world)

Compressive Sensing
(probabilistic world)

Compressive MUSIC
Compressive MUSIC (r=k/2)

Array Signal Processing
(deterministic world)

Compressive Sensing
(probabilistic world)

Compressive MUSIC
Compressive MUSIC (r=k)

Array Signal Processing (deterministic world)  Compressive Sensing (probabilistic world)

Compressive MUSIC
Number of Sensor Elements

- **Partial Support Recovery using SS-OMP**

**Theorem**

(a) \( r \) is a fixed finite number.

(b) Let \( \text{SNR}_{\text{min}}(Y) \) satisfy

\[
\text{SNR}_{\text{min}}(Y) > 1 + \frac{4k}{r} (\kappa(B) + 1).
\]

If we have

\[
m > k(1 + \delta) \left[ 1 - \frac{4k}{r} \frac{(\kappa(B) + 1)}{\text{SNR}_{\text{min}}(Y) - 1} \right]^{-1} \frac{2 \log(n - k)}{r},
\]

then we can find \( k - r \) correct indices of \( \text{supp}X \) by applying subspace S-OMP.
**Partial Support recovery using SS-OMP**

**Theorem**

(a) \( r \) is proportionally increasing with respect to \( k \) so that
\[
\alpha := \lim_{n \to \infty} \frac{r(n)}{k(n)} > 0 \text{ exist.}
\]

(b) Let \( \text{SNR}_{\text{min}}(Y) \) satisfy
\[
\text{SNR}_{\text{min}}(Y) > 1 + \frac{4}{\alpha} (\kappa(B) + 1).
\]

Then if we have
\[
m > k(1 + \delta)^2 \frac{1}{\left[1 - \frac{4}{\alpha} \frac{\kappa(B) + 1}{\text{SNR}_{\text{min}}(Y) - 1}\right]^2 \left[2 - F(\alpha)\right]^2},
\]

for some \( \delta > 0 \) where \( F(\alpha) \) is an increasing function such that \( F(1) = 1 \) and \( \lim_{\alpha \to 0^+} F(\alpha) = 0 \). Then we can find \( k - r \) correct indices of \( \text{supp} X \) by applying subspace S-OMP.
\[ \alpha = \frac{r}{k} \rightarrow 1 \]
Simulation (Noiseless) \( n=100, m=20, r=8 \)

Empirical Recovery Rate

- Compressive MUSIC with 2-thresholding
- Compressive MUSIC with S-OMP
- 2-thresholding
- S-OMP
Simulation (Noiseless)  \( n=100, m=20, r=16 \)
Phase Transition

$r=3$

$r=16$
Joint Sparse Model in DOT

\[ v(x; L) = \int g_0(x, x') u(x'; L) \Delta \mu_a(x') \, dx' \]

scanning

source plane

region of interest

detector plane

unknown value

various illumination patterns
Exact & Non-iterative Reconstruction
(Lee, Ye, 2008, Ye, Bresler, Lee, 2008)

- 1\textsuperscript{st} step : estimate the active index set $\Lambda$ using compressed MUSIC

$$\Lambda = \{ j \in \{1, 2, \ldots, n\} : \Delta \mu_a(x_j) \neq 0 \}$$

- 2\textsuperscript{nd} step : reconstruct the $\Delta \mu_a(x)$

$$\tilde{X} = A_{\Lambda}^\dagger Y \quad \Rightarrow \quad \Delta \tilde{\mu}_a(x_{(j)}) = \frac{\sum_{L=1}^{r} \left( \tilde{u}(x_{(j)}; L) \right)^* \tilde{X}(j, L)}{\sum_{L=1}^{r} |\tilde{u}(x_{(j)}; L)|^2}$$

Foldy-Lax equation

$$u(x_{(j)}; l) = u_0(x_{(j)}; l) - \sum_{i \neq j} g_0(x_{(j)}, x_{(i)}) u(x_{(i)}; l) \Delta \mu_a(x_{(i)})$$

$i, j = 1, \ldots, k, \quad l = 1, 2, \ldots, r_0$
Revisit the Counter Example

\[ \mu_s = 7.5 \text{cm}^{-1} \]
\[ \mu_a = 0.05 \text{cm}^{-1} \]
\[ \Delta \mu_a = 0.05 \text{cm}^{-1} \]

source & detector plane

detector
line source

(a)

(b)

\[ SNR = 40 \text{dB} \]
\[ \leftrightarrow \lambda = 7 \text{mm} \]
\[ \lambda = 5 \text{mm} \]

(c)
Simulation for Molecular Imaging

\[ p(x) = \frac{a(x) \ast a(x)}{a(x) \ast (P_R(Q) - P_R(P_R(Q) A_{I_k-r})a(x))}, \quad x \in \Omega \]

* Generalized MUSIC spectrum

Source and Detector geometry  Original Phantom
<table>
<thead>
<tr>
<th></th>
<th>proposed method</th>
<th>Tikhonov regularization</th>
<th>l1-penalty regularization</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE $[10^{-7}]$</td>
<td>5.8696</td>
<td>7.6837</td>
<td>7.5802</td>
</tr>
<tr>
<td>Hausdorff distance [mm]</td>
<td></td>
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<tr>
<td>Liver</td>
<td>1.5</td>
<td>2.5</td>
<td>2.1794</td>
</tr>
<tr>
<td>Lung (left)</td>
<td>1.118</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Kidney (left)</td>
<td>1</td>
<td>2.4495</td>
<td>2.5495</td>
</tr>
<tr>
<td>Kidney (right)</td>
<td>1.4142</td>
<td>$\infty$</td>
<td>1.8708</td>
</tr>
</tbody>
</table>
Conclusion

- **Diffuse optical tomography** can be formulated **joint sparse recovery problem**
  - Non-iterative and exact reconstruction algorithm exists!

- **Compressive MUSIC** outperforms the all existing methods in joint sparse recovery problems
  - Apply compressive MUSIC for DOT

- Due the **simplicity and effectiveness**, the proposed method would open a new direction in DOT research
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Thank you