Optimal Investment for Worst-Case Crash Scenarios

A Martingale Approach

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Outline

1. Optimal Investment in a Black-Scholes Market
2. Standard Crash Modeling vs. Knightian Uncertainty
3. Worst-Case Optimal Investment
4. Martingale Approach
5. Extensions
In a Black-Scholes market consisting of a riskless bond

\[ dB_t = rB_t dt \]

and a risky asset

\[ dP_t = P_t [(r + \eta) dt + \sigma dW_t] \]

the classical Merton optimal investment problem is to achieve

\[
\max_{\pi} \mathbb{E}[u(X_T^\pi)].
\]

Here \( X = X^\pi \) denotes the wealth process corresponding to the portfolio strategy \( \pi \) via

\[ dX_t = X_t [(r + \pi_t \eta) dt + \pi_t \sigma dW_t], \quad X_0 = x_0, \]

and \( u \) is the investor’s utility for terminal wealth, which we assume to be of the CRRA form \( u(x) = \frac{1}{\rho} x^\rho \), \( x > 0 \), for some \( \rho < 1 \).
Black-Scholes II: Critique

It is well-known that the optimal strategy is to constantly invest the fraction

\[ \pi^* \triangleq \frac{\eta}{(1 - \rho)\sigma^2} \]

of total wealth into the risky asset.

Phenomenon: "Flight to Riskless Assets"

This strategy is not in line with real-world investor behavior or professional asset allocation advice: Towards the end of the time horizon, wealth should be reallocated from risky to riskless investment.
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Black-Scholes III: Crashes

There are two possibilities:

1. Investors and professional consultants are consistently wrong.
2. The model fails to capture an important aspect of reality.

What is the rationale for the behavior described above?

Investors are afraid of a large market crash that has the potential to destroy the value of their stock holdings.
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Crash Modeling II: Jumps in Asset Dynamics

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$$dP_t = P_t[(r + \eta)dt + \sigma dW_t - \ell d\tilde{N}_t]$$

with a compensated Poisson process $\tilde{N}$, then the optimal strategy is

$$\pi^* = \frac{\eta}{(1 - \rho)\sigma^2} + \text{constant correction term}.$$ 

Thus, the effect of a crash is only accounted for ‘in the mean’.
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Unless market crashes depend on the investor’s time horizon, a modification of the asset price dynamics does not resolve the problem.
Recall the intuitive explanation of the phenomenon: Investors are afraid of a major catastrophic event.
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Following F. Knight (1885-1972), let us distinguish two notions of ‘risk’:

- **Risk**: quantifiable, susceptible of measurement, stochastic, statistical, modeled on $(\Omega, \mathcal{F}, \mathbb{P})$
- **Uncertainty**: ‘true’/Knightian/pure uncertainty, no distributional properties, no statistics possible or available
There is ample time series data on regular fluctuations of asset prices, but major crashes are largely unique events. Examples include

- economic or political crises and wars
- natural disasters
- bubble markets
- ... and more.

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Thus, while ordinary price movements are a matter of risk, market crashes are subject to uncertainty.
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Optimal Investment Problem I: Crash Scenarios

We model a financial market crash scenario as a pair

\[(\tau, \ell)\]

where the \([0, T] \cup \{\infty\}\)-valued stopping time \(\tau\) represents the time when the crash occurs, and the \([0, \ell^\infty]\)-valued \(\mathcal{F}_\tau\)-measurable random variable \(\ell\) is the relative crash height:

\[
dP_t = P_t[(r + \eta)dt + \sigma dW_t], \quad P_\tau = (1 - \ell)P_{\tau-}.
\]

Here \(\ell^\infty \in [0, 1]\) is the maximal crash height, and the event \(\tau = \infty\) is interpreted as there being no crash at all.
Optimal Investment Problem II: Portfolio Strategies

The investor chooses a **portfolio strategy** \( \pi \) to be applied before the crash, and a strategy \( \bar{\pi} \) to be applied afterwards.

Given the crash scenario \((\tau, \ell)\), the dynamics of the investor’s **wealth process** \( X = X^{\pi, \bar{\pi}, \tau, \ell} \) are given by

\[
\begin{align*}
dX_t &= X_{t-} [(r + \pi_t \eta) dt + \pi_t \sigma dW_t] \text{ on } [0, \tau), \quad X_0 = x_0, \\
dX_t &= X_{t-} [(r + \bar{\pi}_t \eta) dt + \bar{\pi}_t \sigma dW_t] \text{ on } (\tau, T], \\
X_{\tau} &= (1 - \pi_{\tau}) X_{\tau-} + (1 - \ell) \pi_{\tau} X_{\tau-} = (1 - \pi_{\tau} \ell) X_{\tau-}.
\end{align*}
\]
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$$
\begin{align*}
\text{on } [0, \tau), & \quad X_t = X_0, \\
\text{on } (\tau, T], & \quad X_t = X_{\tau^-} + (1 - \ell)\pi_{\tau}X_{\tau^-} = (1 - \pi_{\tau} \ell)X_{\tau^-}.
\end{align*}
$$

‘High’ values of $\pi$ lead to a high final wealth in the no-crash scenario, but also to a large loss in the event of a crash — ‘low’ values of $\pi$ lead to small or no losses in a crash, but also to a low terminal wealth if no crash occurs.
As above, the investor's attitude towards (measurable, stochastic) risk is modeled by a CRRA utility function

\[ u(x) = \frac{1}{\rho} x^\rho, \quad x > 0, \quad \text{for some } \rho < 1. \]

By contrast, he takes a worst-case attitude towards the (Knightian, ‘true’) uncertainty concerning the financial market crash, and thus faces the

**Worst-Case Optimal Investment Problem**

\[
\max_{\pi, \bar{\pi}} \min_{\tau, \ell} \mathbb{E}[u(X_T^{\pi, \bar{\pi}, \tau, \ell})]. \tag{P}
\]

Problem (P) reflects an extraordinarily cautious attitude towards the threat of a crash. Note that there are no distributional assumptions on the crash time and height. Observe also that portfolio strategies are not compared scenario-wise.
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Martingale Approach I: Idea and Motivation

The fundamental ideas underlying the martingale approach to worst-case optimal investment are:

- The worst-case investment problem can be regarded as a **game** between the investor and the market.
- The notion of **indifference** plays a fundamental role in this game.
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The fundamental ideas underlying the martingale approach to worst-case optimal investment are:

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- The notion of *indifference* plays a fundamental role in this game.

The martingale approach consists of 3 main components:

- the **Change-of-Measure Device**,  
- the **Indifference-Optimality Principle**, and  
- the **Indifference Frontier**.
To solve the post-crash portfolio problem, we use a well-known trick:

**Theorem (Change-of-Measure Device)**

Consider the classical optimal portfolio problem with random initial time $\tau$ and time-$\tau$ initial wealth $\xi$,

$$\max_{\bar{\pi}} \mathbb{E}[u(X_{\bar{\pi}T}) \mid X_{\bar{\pi}\tau} = \xi]. \quad (P_{\text{post}})$$

Then for any strategy $\bar{\pi}$ we have

$$u(X_{\bar{\pi}T}) = u(\xi) \exp\left\{ \rho \int_{\tau}^{T} \Phi(\bar{\pi}_s) ds \right\} M_{\bar{\pi}T}$$

with $\Phi(y) \triangleq r + \eta y - \frac{1}{2}(1 - \rho)\sigma^2 y^2$ and a martingale $M_{\bar{\pi}T}$ satisfying $M_{\bar{\pi}\tau} = 1$. Thus the solution to $(P_{\text{post}})$ is the Merton strategy $\pi^M$. 

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Post-Crash Problem II: Reformulation

The Change-of-Measure Device allows us to reformulate the worst-case investment problem (P)

$$\max_{\pi, \bar{\pi}} \min_{\tau, \ell} \mathbb{E}[u(X^{\pi, \bar{\pi}, \tau, \ell}_T)]$$

as the

Pre-Crash Investment Problem

$$\max_{\pi} \min_{\tau} \mathbb{E}[V(\tau, (1 - \pi_{\tau, \ell}^{\infty})X^{\pi}_T)]. \quad (P_{\text{pre}})$$

Here $V$ is the value function of the post-crash problem,

$$V(t, x) = \exp\{\rho \Phi(\pi^M)(T - t)\} u(x).$$
Controller-vs-Stopper I: Abstract Formulation

The formulation \( (P_{pre}) \) takes the form of the abstract

**Controller-vs-Stopper Game [Karatzas and Sudderth (2001)]**

Consider a zero-sum stochastic game between player \( A \) (the controller) and player \( B \) (the stopper). Player \( A \) controls a stochastic process

\[
W = W^\lambda \text{ on the time horizon } [0, T]
\]

by choosing \( \lambda \), and player \( B \) decides on the duration of the game by choosing a \([0, T] \cup \{\infty\}\)-valued stopping time \( \tau \). The terminal payoff is \( W^\lambda_\tau \). Thus player \( A \) faces the problem

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\max_\lambda \min_\tau \mathbb{E}[W^\lambda_\tau]. \quad (P_{abstract})
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\[ \max_\lambda \min_\tau \mathbb{E}[W^\lambda_\tau]. \]  

(P_{abstract})

In the worst-case investment problem,

\[ W^\lambda_t = V(t, (1 - \pi_t \ell^\infty) X^\pi_t), \quad t \in [0, T], \quad W^\lambda_\infty = V(T, X^\pi_T) = u(X^\pi_T). \]
Controller-vs-Stopper II: Indifference-Optimality Principle

If player $A$ can choose his strategy $\hat{\lambda}$ in such a way that $\mathcal{W}^{\hat{\lambda}}$ is a martingale, then player $B$’s actions become irrelevant to him:

$$
\mathbb{E}[\mathcal{W}^{\hat{\lambda}}_{\sigma}] = \mathbb{E}[\mathcal{W}^{\hat{\lambda}}_{\tau}] \text{ for all stopping times } \sigma, \tau.
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Hence, we say that $\hat{\lambda}$ is an (abstract) indifference strategy.
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Hence, we say that $\hat{\lambda}$ is an (abstract) indifference strategy.

**Proposition (Indifference-Optimality Principle)**

If $\hat{\lambda}$ is an indifference strategy, and for all $\lambda$ we have

$$ \mathbb{E}[W^{\hat{\lambda}}_\tau] \geq \mathbb{E}[W^{\lambda}_\tau] \text{ for just one stopping time } \tau, $$

then $\hat{\lambda}$ is optimal for player $A$ in $(P_{\text{abstract}})$. 
Indifference I: Indifference Strategy

The indifference strategy \( \hat{\pi} \) for worst-case investment is given by the o.d.e.

\[
\dot{\hat{\pi}}_t = -\frac{\sigma^2}{2\ell_\infty} (1 - \rho)[1 - \hat{\pi}_t \ell_\infty][\hat{\pi}_t - \pi^M]^2, \quad \hat{\pi}_T = 0. \tag{I}
\]

The indifference strategy is below the Merton line and satisfies \( \hat{\pi}_t \ell_\infty \leq 1 \). It converges towards the Merton strategy if \( \pi^M \ell_\infty \leq 1 \).
Indifference II: Indifference Frontier

The indifference strategy represents a **frontier** which rules out too naïve investment.

**Lemma (Indifference Frontier)**

Let $\hat{\pi}$ be determined from (I), and let $\pi$ be any portfolio strategy. Then the worst-case bound attained by the strategy $\tilde{\pi}$,

$$\tilde{\pi}_t \triangleq \pi_t \text{ if } t < \sigma, \quad \tilde{\pi}_t \triangleq \hat{\pi}_t \text{ if } t \geq \sigma,$$

where $\sigma \triangleq \inf\{t : \pi_t > \hat{\pi}_t\}$, is at least as big as that achieved by $\pi$. 
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where $\sigma \triangleq \inf\{t : \pi_t > \hat{\pi}_t\}$, is at least as big as that achieved by $\pi$.

**Proof.**

Since $W_{\tilde{\pi}} = W_{\hat{\pi}}$ is a martingale for $t > \sigma$ and $W_{\tilde{\pi}} = W_{\pi}$ for $t \leq \sigma$,

$$\mathbb{E}[W_{\tilde{\pi}}] = \mathbb{E}[W_{\hat{\pi}}^\pi] = \mathbb{E}[W_{\pi}^\pi] \geq \min_{\tau'} \mathbb{E}[W_{\tau'}^\pi]$$

for an arbitrary stopping time $\tau$.  

$\blacksquare$
Combining the previous results, we arrive at the following

**Theorem (Solution of the Worst-Case Investment Problem)**

For the worst-case portfolio problem

\[
\max_{\pi, \tilde{\pi}} \min_{\tau, \ell} \mathbb{E}[u(X_T^{\pi, \tilde{\pi}, \tau, \ell})] \quad (P)
\]

the optimal strategy in the pre-crash market is given by the indifference strategy \( \hat{\pi} \). After the crash, the Merton strategy \( \pi^M \) is optimal.
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**Theorem (Solution of the Worst-Case Investment Problem)**

For the worst-case portfolio problem

$$\max_{\pi, \bar{\pi}} \min_{\tau, \ell} \mathbb{E}[u(X^{\pi, \bar{\pi}, \tau, \ell}_T)]$$  \hspace{1cm} (P)

the optimal strategy in the pre-crash market is given by the indifference strategy \( \hat{\pi} \). After the crash, the Merton strategy \( \pi^M \) is optimal.

**Proof.**

We need only consider pre-crash strategies below the Indifference Frontier. By the Indifference-Optimality Principle, the indifference strategy is optimal provided it is optimal in the no-crash scenario. This, however, follows immediately from the Change-of-Measure Device.
To illustrate the difference to traditional portfolio optimization, we determine the **effective wealth loss** of a Merton investor in his worst-case scenario.
Solution IV: Sensitivity to Crash Size

The solution to the worst-case investment problem is non-zero even for a maximum crash height $\ell^\infty = 100\%$. 
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The martingale approach generalizes directly to **multi-asset markets**. In this multi-dimensional setting, the indifference frontier is specified by

\[ \pi \ell^\infty \leq \hat{\beta}_t. \]

where \( \hat{\beta} \) is characterized by a one-dimensional o.d.e.
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where \( \hat{\beta} \) is characterized by a one-dimensional o.d.e.
Regular price jumps can be included in the stock price dynamics; thus the investor distinguishes regular jumps (risky) from crashes (uncertain).

\[
dP_t = P_{t-} \left[ (r + \eta)dt + \sigma.dW_t - \int \xi \nu(dt, d\xi) \right], \quad P_\tau = (1 - \ell)P_{\tau-}.
\]
Alternative Dynamics I: Regular Jumps

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\[
dP_t = P_{t-} \left[ (r + \eta)dt + \sigma dW_t - \int \xi \nu(dt, d\xi) \right], \quad P_T = (1 - \ell)P_{T-}.
\]

The effects are similar to the Black-Scholes case:
We can model different market regimes by allowing the market coefficients to change after a possible crash:

\[
\begin{align*}
\text{d}P_t &= P_t \left[ (r + \eta) \text{d}t + \sigma \text{d}W_t \right] \text{ on } [0, \tau) \\
\text{d}P_t &= P_t \left[ (\bar{r} + \bar{\eta}) \text{d}t + \bar{\sigma} \text{d}\bar{W}_t \right] \text{ on } [\tau, T], \quad P_\tau = (1 - \ell)P_\tau.
\end{align*}
\]
Alternative Dynamics II: Regime Shifts

We can model different market regimes by allowing the market coefficients to change after a possible crash:

\[
\begin{align*}
\mathrm{d}P_t &= P_t^- [(r + \eta) \mathrm{d}t + \sigma \mathrm{d}W_t] \text{ on } [0, \tau) \\
\mathrm{d}P_t &= P_t^- [({\bar{r}} + \bar{\eta}) \mathrm{d}t + \bar{\sigma} \mathrm{d}{\bar{W}}_t] \text{ on } [\tau, T], \quad P_T = (1 - \ell)P_T.
\end{align*}
\]

Now we need to distinguish between bull and bear markets:

If the post-crash market is worse than the pre-crash riskless investment, the investor perceives a bear market; in this case, it is optimal not to invest in risky assets.
Alternative Dynamics III: Bull Markets

On the other hand, in a **bull market** it is optimal to use the indifference strategy as long as it is below the Merton line:
Finally the model can be extended to multiple crashes. The worst-case optimal strategy can be determined by backward recursion:
Thank you very much for your attention!