Portfolio insurance under risk-measure constraint

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6th World Congress of the Bachelier Finance Society
June 22-26 2010

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The Problem
Decoupling
Examples

The Insurance

Investor
Has utility function U
Wants a guaranteed return z

Fund manager
Maximizes $U((X_T-z)^+)$
subject to
$\rho(-(X_T-z)^-) \leq \rho_0$

Bank
Guarantees the fund
Wants to limit exposure

$X_0$, $X_T$
Market assumptions

We will assume that:

- The market is complete with a unique martingale measure $\xi P$ on $(\Omega, \mathcal{F})$
- The risk is measured in terms of a law-invariant convex risk measure $\rho$ continuous from above.

$$\rho (X) := \sup_{Q \in \mathcal{M}_1 (P)} (\mathbb{E}_Q [-X] - \gamma_{min} (Q))$$

we will suppose $\rho (0) = 0$

- The risk exposure imposed on the Fund manager is given by $\rho_0$
If we let

\[ H := \left\{ X \in L^1(P) \mid \mathbb{E}[\xi X] \leq x_0, 0 \leq \rho \left( - (X - z)^- \right) \leq \rho_0 \right\} \]

then the FM’s aim is to find, if it exists, a \( X^* \in H \) such that:

\[ \mathbb{E} \left[ u (X^* - z)^+ \right] = \sup_{X \in H} \mathbb{E} \left[ u (X - z)^+ \right] \]

and the optimal payoff for the Investor will be

\[ \max (X^*, z) \]
Define $U(X) := \mathbb{E}[u((X - z)^+)]$ and remark that

$$U(X) = U(X1_A)$$

where $A := \{X \geq z\}$. This means that only $X1_A$ remains important for the investor. This remark suggests this decoupling:
Decoupling-Idea

let \((A, x^+) \in \mathcal{F} \times \mathbb{R}^+\) and

\[
\begin{align*}
\mathcal{P}_1 : & \quad \sup_{X} U(X) \quad \text{s.t.} \\
& \quad \mathbb{E}[\xi X] \leq x^+, \quad X \in L^1(\mathbb{P}) \text{ and} \\
& \quad X = 0 \quad \text{on} \ A_c, \quad X \geq z \quad \text{on} \ A
\end{align*}
\]

and

\[
\begin{align*}
\triangle (A) : & \quad \inf_{Y} \mathbb{E}[\xi Y] \quad \text{s.t.} \\
& \quad \rho \left( - (Y - z)^- 1_{A^c} \right) \leq \rho_0, \quad Y \in L^1(\mathbb{P}) \text{ and} \\
& \quad Y = 0 \quad \text{on} \ A, \quad Y \leq z \quad \text{on} \ A^c
\end{align*}
\]

Define also \(x_+(A) := x_0 - \triangle (A)\). Remark upon how both these problems can be solved by Lagrangian methods.
Decoupling-Idea

The next example will clarify the role of $\triangle (A)$. Fix $A$ such that $0 < P(A) < 1$ and suppose $\triangle (A) = -\infty$. It is possible to find, $\forall n \in \mathbb{N}$ a $Y^n \in \mathcal{P}_2 (A)$ such that $\mathbb{E} [\xi Y^n] \leq -n$. Consider now this payoff

$$X^n = \frac{x_0 + n}{\mathbb{E} [\xi 1_A]} 1_A + Y^n$$

We deduce $X^n \in H$ and $U(X^n) \to +\infty$, which means that our problem has no finite solution.
We will then carry out the following:

Assumption

$$\inf_{A \in \mathcal{F}} \triangle (A) > -\infty$$
The following condition guarantees our assumption:

**Theorem**

Let \( \rho \) be a law-invariant convex risk measure and \( \xi \) the risk-neutral probability of the market. If

\[
\gamma_{\min}(\xi^\mathbb{P}) < +\infty
\]

then \( \inf_A \triangle (A) > -\infty \).
Let $X (A, x^+)$ the solution of problem $\mathcal{P}_1$ with parameters $A$ and $x^+$ and recall that $x^+ (A) := x_0 - \triangle (A)$

**Theorem**

If $\inf_A \triangle (A) > -\infty$ then

$$\sup_{X \in H} U (X) = \sup_{A \in \mathcal{F}} U (X (A, x^+ (A)))$$

If $\inf_A \triangle (A) = -\infty$ then we already know

$$\sup_{X \in H} U (X) = +\infty$$
Using the last Theorem, we can solve our problem as the following:

1. fix $A \in \mathcal{F}$
2. solve $\mathcal{P}_2(A)$ and find $\triangle(A)$
3. solve $\mathcal{P}_1(A)$ with parameter $x^+(A)$
4. maximize the value function of $\mathcal{P}_1(A)$, $U(X(A, x^+(A)))$, over $A \in \mathcal{F}$
We can use the last result to give a necessary and sufficient condition for the existence of a finite solution.

**Theorem**

Assume $\inf_A \triangle (A) > -\infty$ and $X^*$ is optimal for our problem. Define $A^* := \{X^* \geq z\}$. One has

$$\sup_{A \in \mathcal{F}} U \left( X \left( A, x^+ (A) \right) \right) = U \left( X \left( A^*, x^+ (A^*) \right) \right)$$

$$\triangle (A^*) = \mathbb{E} [\xi Y^*], \text{ where } Y^* := X^* - X^* 1_{A^*}$$
Reciprocally, let $A^* \in \mathcal{F}$ and a $Y^* \in \mathcal{P}_2(A^*)$ such that

$$U\left(X\left(A^*, x^+(A^*)\right)\right) = \sup_{A \in \mathcal{F}} U\left(X\left(A, x^+(A)\right)\right)$$

$$\mathbb{E}[\xi Y^*] = \triangle(A^*) = \inf_{Y \in \mathcal{H}_2(A^*)} \mathbb{E}[\xi Y]$$

Then a solution of our problem is given by

$$X^* := X\left(A^*, x^+(A^*)\right)1_{A^*} + Y^*1_{A^*,c}$$

In this case, the payoff for the investor will be

$$\text{Payoff} = X\left(A^*, x^+(A^*)\right)1_{A^*} + z$$
Generally a maximization over the sets in \( \mathcal{F} \) is not simple.

Our aim here is to show that this latter maximization may be carried out over a subset of \( \mathcal{F} \), parameterized by a real number, Jin and Zhou (2008). Define

\[ v(A) := \sup_{X \in P_1(A, x^+(A))} U(X) \]

so then

\[ \sup_{X \in H} U(X) = \sup_{A \in \mathcal{F}} U(X(A, x^+(A))) = \sup_{A \in \mathcal{F}} v(A) \]
Theorem

Suppose $\xi$ has not atoms. Define $\underline{\xi} := \text{essinf} \xi$ and $\overline{\xi} := \text{esssup} \xi$. Let $A \in \mathcal{F}$ and $c \in [\underline{\xi}, \overline{\xi}]$ such that $\mathbb{P}(\xi \leq c) = \mathbb{P}(A)$. Then

$$v(A) \leq v(\{\xi \leq c\})$$

which means

$$\sup_{X \in H} U(X) = \sup_{A \in \mathcal{F}} v(A) = \sup_{c \in [\underline{\xi}, \overline{\xi}]} v(\{\xi \leq c\})$$
Using the last Theorem we can solve our problem as the following:

1. fix $c \in [\xi, \xi]$
2. solve $P_2 (c)$ and find $\triangle (c)$
3. solve $P_1 (c)$ with parameter $x_+ (c) = x_0 - \triangle (c)$
4. find $c^*$ that maximizes $U \left( X_1 \left( \{\xi \leq c\} \,, \, x_+ (c) \right) \right)$
5. A optimal payoff for the Investor will be $X^* = X_1 \left( \{\xi \leq c\} \,, \, x_+ (c) \right) 1_{\{\xi \leq c\}} + Z$
We will now see what happens when \( \rho = CVaR_\lambda, \lambda \in (0, 1) \):

\[
CVaR_\lambda (X) := \frac{1}{\lambda} \int_0^\lambda \text{VAR}_u (X) \, du
\]

or, equivalently

\[
CVaR_\lambda (X) = \int_0^{+\infty} \psi_\lambda (\mathbb{P} (-X > t)) \, dt
\]

where

\[
\psi_\lambda (u) = \frac{(u \wedge \lambda)}{\lambda}
\]
We then have the following:

**Theorem**

Let $\xi$ the state price density.

i) If $\xi$ is unbounded then our problem has no finite solution

ii) If $\xi$ is bounded then our value function is:

$$
\sup_{X \in H} U(X) = \sup_{c \in [\xi, \bar{\xi}]} \mathbb{E} \left[ u \left( \lambda(c, \xi) \right) \right] 1_{\{\xi \leq c\}}
$$
Example-CVaR

where

- \( I = (u')^{-1} \)
- \( \lambda(c) \) is given by: \( \mathbb{E} \left[ \xi \left( [I(\lambda(c)\xi)]^+ \right) 1_{\{\xi \leq c\}} \right] = x_0 + \rho_0 \beta \xi \)

We do not have a solution for the Fund Manager problem because problem \( \mathcal{P}_2 \) does not have a minimum. However we can give a solution for the investor which is

\[
X^* = z + [I(\lambda(c^*)\xi)]^+
\]
Note also that the minimal penalty function for the $CVaR_\lambda$ is given by:

$$
\gamma_{min}(Q) := \begin{cases} 
0 & \text{if } \frac{dQ}{dP} \leq \frac{1}{\lambda}, \quad P\text{-a.s} \\
+\infty & \text{otherwise}
\end{cases}
$$

So, for example, if we have $\xi$ bounded but $P(\xi > \frac{1}{\lambda}) > 0$ then it turns out $\gamma_{min}(\xi P) = +\infty$ even if the problem has a solution!

Here is a good example where we have a solution even if $\gamma_{min}(\xi P) = +\infty$!
If we consider $\rho = ERM_\lambda$, where $\lambda > 0$ and

$$ERM_\lambda (X) := \lambda \ln \mathbb{E} \left[ \exp \left( -\frac{1}{\lambda} X \right) \right]$$

We have:
Theorem

Assume that the state price density $\xi$ has no atoms and satisfies $\xi \log \xi \in L^1(\mathbb{P})$. Then the optimal payoff for the fund manager is given by

$$X^* := z + [I(\lambda(c^*) \xi)]^+ 1_{\{\xi \leq c^*\}} - \beta \left[ \log \left( \frac{\beta}{\eta(c^*) \xi} \right) \right]^+ 1_{\{\xi > c^*\}}$$
Example-Entropic Risk Measure

where

- \( I = (u')^{-1} \)
- \( \lambda (c) \) is given by: \( E[\xi [I(\lambda(c)\xi)]^+ \mathbf{1}_{\{\xi \leq c\}}] = x_0 - \Delta (c) \)
- \( \alpha (c) = P(\xi > c) \)
- \( \psi (c) := E[\xi \mathbf{1}_{\{\xi > c\}}] \)
- \( \Delta (c) = -\beta \left( \log \left( \frac{\beta}{\eta(c)} \right) \psi \left( c \vee \frac{\eta(c)}{\beta} \right) + \hat{\psi} \left( c \vee \frac{\eta(c)}{\beta} \right) \right) \)
- \( \eta (c) \) is given by: \( \frac{\beta}{\eta(c)}\psi \left( c \vee \frac{\eta(c)}{\beta} \right) + P \left( c < \xi \leq \frac{\eta(c)}{\beta} \right) = e^{\rho_0/\beta} + \alpha (c) - 1 \)
- \( c^* \) attains the supremum of \( c \rightarrow E[u ([I(\lambda(c)\xi)]^+) \mathbf{1}_{\{\xi \leq c\}}] \)
Again, the proof is not complicated; one just needs to follow the Algorithm 2.

Remark that the penalty function for the $ERM_\lambda$:

$$\gamma_{min} (Q) := \lambda H (Q \mid \mathbb{P}) := \lambda \mathbb{E}_Q \left[ \log \left( \frac{dQ}{d\mathbb{P}} \right) \right]$$

With our hypothesis, we easily have $\gamma_{min} (\xi \mathbb{P}) < \infty$: we know that this is a sufficient condition under which the problem has a solution. The condition $\xi \log \xi \in L^1 (\mathbb{P})$ is naturally verified in a Black-Scholes framework.
We will see now what happens in a very simple one-dimensional Black-Scholes model: On \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), let

\[
dS_t = S_t \left( bt + \sigma dW_t \right)
\]

and suppose \(\mu = b/\sigma > 0\). The unique equivalent martingale measure is given by \(\mathbb{Q} = \xi \mathbb{P}\), where

\[
\xi = \exp(-\mu W_T - \mu^2 T/2) = \left[ S_T \exp \left( T \left( \sigma^2 - b \right)/2 \right) / S_0 \right]^{-\frac{b}{\sigma^2}}.
\]
Numerical Results

We will use the utility function $u(x) = 1 - e^{-\delta x}$ and the $ERM_{\lambda}$ as risk measure. Our initial data is:

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<th>Data</th>
<th></th>
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<tr>
<td>drift</td>
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<tr>
<td>volatility</td>
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<tr>
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<tr>
<td>entropic constant ($\lambda$)</td>
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</tr>
<tr>
<td>utility constant ($\delta$)</td>
<td>0.6</td>
</tr>
</tbody>
</table>
An optimal payoff will be:

\[
X^* := \left[ \frac{L}{\delta} \log (S_T) + K_1 \right]^+ \mathbf{1}_{\{S_T \geq s^\star\}} - \beta \left[ K_2 - L \log (S_T) \right]^+ \mathbf{1}_{\{S_T < s^\star\}} + z
\]

where

\[
s^\star = 0.9375, \quad K_1 = 1.34026, \quad K_2 = 3.18886
\]

Other quantities one can also compute are optimal \( c^\star \), value functions of problems \( \mathcal{P}_1 - \mathcal{P}_2 \) and the "success" probability:

\[
c^\star = 2.72293, \quad \nu (c^\star) = 0.900134
\]
\[
\triangle (c^\star) = -1.17387, \quad \mathbb{P} (S_T \geq s^\star) = 0.946722
\]
The following figure is the value function \( c \rightarrow v(c) \): 

![Value Function Graph](image-url)
The Payoff profile for the Fund Manager

The payoff profile for the fund manager
Suppose, for sake of simplicity, $z = 0$ and let us see what happens if we do not allow any risk, i.e. $\rho_0 = 0$. We can see this by solving the following problem

$$\max \mathbb{E}[1 - e^{-\delta X^+}]$$

$$\mathbb{E}[X] \leq x_0, \quad X \geq 0$$

and compare the payoff profiles
The Problem
Decoupling
Examples
Numerical Results

Graphics

Payoff profile in case of risk constraint vs Payoff profile with the condition X>0

The payoff profile in case of risk constraint
The payoff profile under the condition X>0

Portfolio insurance under risk-measure constraint