Simple Improvement Method for Upper Bound of American Option

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6th World Congress of the Bachelier Finance Society
June, 2010@Hilton, Toronto, Canada
Numerical Methods for Pricing American Option

1. **Closed-Form Solution**: It is difficult to find a closed-form solution.

2. **Lattice Methods**: When the condition is simple, the lattice methods give good approximated solutions.

3. **Monte Carlo Simulation**: When the condition is complicated, the Monte Carlo simulation is practical.

**Monte Carlo simulation**

- **Lower Bound**: A stopping time gives a lower bound.
  - The least-square method gives a good stopping time. Longstaff and Schwartz (2001)
- **Upper Bound**: A martingale gives an upper bound.
  - Can we find a good martingale?
Setup

The saving account is the numeraire. All prices are discounted prices.

\( T \in \mathbb{N} \) : Fixed Maturity
\((\Omega, \mathcal{F}, P, \{\mathcal{F}_k; k = 0, 1, \ldots, T\})\) : Filtered probability space
\( S_k (k = 0, 1, \ldots, T) \) : Price Process of Risky Asset
\( H_k (k = 0, 1, \ldots, T) \) : Payoff of American Option
\( V_k (k = 0, 1, \ldots, T) \) : Price of American Option

Assumption

- \( P \) is a unique equivalent martingale measure.
- \( \mathcal{F}_k \) is a natural filtration generated by \( S \). We write \( E_k[\cdot] = E[\cdot|\mathcal{F}_k] \).
- \( H \) is an adapted process.

Definition 1 A supersolution is a supermartingale \( X \) satisfying

\[ X_k \geq H_k, \quad k = 0, 1, \ldots, T - 1 \]

and the maturity condition, that is, \( X_T = H_T \).

\( V \) is a minimum supersolution.

\( \triangleright \) Any supersolution is an upper bound process of the American option.
Main Problem

Suppose that a supersolution \( U \) is given. Note that \( U_0 \) is an upper bound.
Suppose that the lower bound process \( L \) of the continuation value is given.

\[
L_k \leq E_k[V_{k+1}] \leq V_k \leq U_k, \quad k < T,
\]

continuation value

\[
L_T = H_T (= U_T).
\]

We want to improve the upper bound \( U_0 \) in the Monte Carlo simulation.
Chen and Glasserman (2007) proposes an iterative method.

1. Using the supersolution \( U \), a martingale is given by
   \[
   M_k^U = \sum_{t=1}^{k} (U_t - E_{t-1}[U_t]), \quad k = 0, 1, \ldots, T.
   \]

2. Using the martingale \( M \), a new supersolution (= upper bound process) is
given by \( U_k^M = E_k[\max_{k \leq t \leq T}(H_t - M_t)] + M_k, \quad k = 0, 1, \ldots, T. \)

- The iterative improvement converges to the true price.
- The calculation of the conditional expectation is necessary at all times
  and all states for the Doob decomposition.
- The lower bound process is not used.

We want to find a computationally-efficient improvement method using \( L \).
Basic Result

Let $\mathcal{T}^k$ be the set of the stopping times whose values are greater than or equal to $k$.

**Theorem 1** Let $\tau_1, \tau_2 \in \mathcal{T}^0$ and $\tau_1 \leq \tau_2$. Suppose that $V$ satisfies the martingale property in $[0, \tau_1] \cup [\tau_1 + 1, \tau_2]$, that is,

$$V_k = E_k[V_{k+1}], \quad k \in [0, \tau_1 - 1] \cup [\tau_1 + 1, \tau_2 - 1].$$

Let

$$w(\tau_1, \tau_2) = E[\max(H_{\tau_1}, E_{\tau_1}[U_{\tau_2}])].$$

Then

$$V_0 \leq w(\tau_1, \tau_2) \leq U_0.$$  

The problem is to find an appropriate pair of stopping times $(\tau_1, \tau_2)$. 

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**New Upper Bound**
Methods 1, 2

We use the mathematical convention the minimum over the empty set is $\infty$,
\[ \min(\emptyset) = +\infty. \]

**Lemma 1** Let $\tau_1^* = \min\{k \geq 0 | H_k > L_k\} \wedge T$. Then $V$ satisfies the martingale property in $[0, \tau_1^*]$, that is, $V_k = E_k[V_{k+1}]$ for $k \in [0, \tau_1^* - 1]$.

**Corollary 1** Let $w_L^1 = w(\tau_1^*, \tau_1^*)$. Then $V_0 \leq w_L^1 \leq U_0$.

**Corollary 2** Let $w_L^2 = w(\tau_1^*, (\tau_1^* + 1) \wedge T)$. Then $V_0 \leq w_L^2 \leq w_L^1 \leq U_0$.

- $w_L^2 \leq w_L^1$. \cdots $w_L^2$ is a better upper bound than $w_L^1$.
- When $U_k = E_k[\max_{k \leq t \leq T}(H_t - M_t)] + M_k$,
  \[ w_L^1 = E[\max_{\tau_1^* \leq t \leq T}(H_t - M_t)] , \]
  \[ w_L^2 = E[\max_{\tau_1^*} (H_{\tau_1^*}, E_{\tau_1^*} [\max_{(\tau_1^* + 1) \wedge T \leq t \leq T}(H_t - M_t)] + M_{\tau_1^*})]. \]
  - $w_L^1$ includes no conditional expectation per path.
  - $w_L^2$ requires only one conditional expectation per path.
  - The iterated method requires $T$ conditional expectations per path.

The calculations of $w_L^1$ and $w_L^2$ spend much less time than that of the iterative method. The proposed methods are more efficient.
Method 3

Lemma 2 Let \( \tau_2^* = \min\{k > \tau_1^* | H_k > L_k\} \wedge T \). Then \( V \) satisfies the martingale property in \([\tau_1^* + 1, \tau_2^*]\), that is, \( V_k = E_k[V_{k+1}] \) for \( k \in [\tau_1^* + 1, \tau_2^* - 1] \).

Corollary 3 Let

\[
\mathcal{W}_L^3 = w(\tau_1^*, \tau_2^*).
\]

Then

\[
V_0 \leq \mathcal{W}_L^3 \leq \mathcal{W}_L^2 \leq U_0.
\]

- \( \mathcal{W}_L^3 \) is the best upper bound of the three proposed methods.
- When \( U_k = E_k[\max_{k \leq t \leq T} (H_t - M_t)] + M_k \),

\[
\mathcal{W}_L^3 = E[\max \left( H_{\tau_1^*}, E_{\tau_1^*}[\max_{\tau_2^* \leq t \leq T} (H_t - M_t)] + M_{\tau_1^*} \right)].
\]

We have to calculate \( \tau_2^* \). When the lower bound process can be calculated by an analytic formula, the calculation of \( \tau_2^* \) is not time-consuming and then the amount of calculation of \( \mathcal{W}_L^3 \) is as much as that of \( \mathcal{W}_L^2 \).
Lower Bound Effect

**Lemma 3** Let $\tau_a, \tau_b \in T^0$. If $\tau_a \leq \tau_b$, then

\[
\begin{align*}
    w(\tau_a, \tau_a) &\geq w(\tau_b, \tau_b), \\
    w(\tau_a, (\tau_a + 1) \land T) &\geq w(\tau_b, (\tau_b + 1) \land T).
\end{align*}
\]

**Proposition 1** Let $L^a$ and $L^b$ be lower bound processes. Suppose that

\[
L^a_k \leq L^b_k, \quad k = 0, 1, \ldots, T. \quad L^b \text{ is a better lower bound process than } L^a.
\]

Then

\[
\begin{align*}
    w^1_{L^a} &\geq w^1_{L^b}, \\
    w^2_{L^a} &\geq w^2_{L^b}, \\
    w^3_{L^a} &\geq w^3_{L^b}.
\end{align*}
\]

The better a lower bound process is, the greater improvement of upper bound can be expected.
European Option Based Model

Let $V^E$ be the price process of the European option satisfying $V^E_k = E_k[H_T]$.

$$M_k = V^E_k - V^E_0,$$
$$U_k = E_k[\max_{k \leq t \leq T} (H_t - M_t)] + M_k.$$

We call this model the European option based model.

**Proposition 2** Consider the European option based model with $L = V^E$. If $\tau \in T^0$ satisfies $\tau < \tau_1^*$, then

$$U_0 = w(\tau, \tau) = w(\tau, (\tau + 1) \land T).$$

If $L$ is smaller than $V^E$, it fails to improve the upper bound.

**Proposition 3** In the European option based model, if $L = V^E$, then we have

$$U_0 = w^1_L \geq w^2_L = w^3_L.$$

- $V^E$ is the worst lower bound which may improve the upper bound.
- We check whether $w^2_L = w^3_L$ generated by $V^E$ can improve the upper bound by the numerical analysis.
Simulation Condition

- The price process is given by the Black Scholes Model, that is,

\[ S_k = S_{k-1} \exp \left( -\frac{\sigma^2}{2} \Delta t + \sigma \sqrt{\Delta t} \xi_k \right), \quad k = 1, \ldots, T, \]

\[ H_k = \max \left( Ke^{-r_k \Delta t} - S_k, 0 \right), \quad k = 0, 1, \ldots, T, \]

where \( \xi_1, \ldots, \xi_T \) are independent and standard normally distributed.

- Let \( L = V^E \), that is,

\[ L_k = K \Phi(d(k, T, K, 0)) - S_k \Phi(d(k, T, K, \sigma^2)), \quad k = 0, 1, \ldots, T - 1 \]

where \( \Phi(\cdot) \) is the standard normal distribution function and

\[ d(k, T, K, r) = \frac{1}{\sigma \sqrt{(T - k)\Delta t}} \left( \log \frac{K}{S_k} - \left( r - \frac{1}{2} \sigma^2 \right)(T - k)\Delta t \right). \]

- \( S_0 = 100, \quad r = 0.04, \quad \sigma = 0.3, \quad \Delta t = 0.01, \quad T = 50, 100, 150. \)
- The number of paths for calculating the expectation is 2,500.
- The number of paths for calculating the conditional expectation is 500.
- The antithetic sampling is used.
Better Lower Bound

• Let $L_T^a = L_T^b = H_T$ and for $k = 0, 1, \ldots, T - 1,$

$$L_k^a = \max_{t_0 > k} \left( \sup_{\tau \in \mathcal{T}_{t_0}, \tau} E_k[H_\tau] \right), \quad L_k^b = \sup_{\tau \in \mathcal{T}_{k+1}} E_k[H_\tau]$$

where $\mathcal{T}_{t_0, \tau}$ is the set of the stopping times whose values are $t_0$ or $\tau$.

• $L^a$ can be calculated by the analytic formula since

$$\sup_{\tau \in \mathcal{T}_{t_1}, \tau} E_{t_0}[H_\tau] = K\Phi(d(t_0, t_1, S_{t_1}^*, 0)) - S_{t_0}\Phi(d(t_0, t_1, S_{t_1}^*, \sigma^2))$$

$$+ K\Phi_2(-d(t_0, t_1, S_{t_1}^*, 0), d(t_0, T, K, 0); \frac{t_1 - t_0}{T - t_0})$$

$$- S_{t_0}\Phi_2(-d(t_0, t_1, S_{t_1}^*, \sigma^2), d(t_0, T, K, \sigma^2); \frac{t_1 - t_0}{T - t_0})$$

where $\Phi_2(\cdot, \cdot; \rho)$ is the standard bivariate normal distribution function. $S_{t_1}^*$ is a solution of

$$K\Phi(d(t_1, T, K, 0)) - S_{t_1}^*\Phi(d(t_1, T, K, \sigma^2)) = Ke^{-rt_1\Delta t} - S_{t_1}^*.$$

• $L^b$ is used in order to estimate the maximum improvement.

Note that $L^b$ can be calculated by the lattice tree.
# Numerical Result (Lower Bound Effect)

### $K = 90$ (OTM)

<table>
<thead>
<tr>
<th>$T$</th>
<th>$U_0$</th>
<th>$w_L^3$</th>
<th>$w_{La}^3$</th>
<th>$w_{Lb}^3$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>3.471(0.002)</td>
<td>3.469(0.002)</td>
<td>3.465(0.002)</td>
<td>3.463(0.002)</td>
<td>3.460</td>
</tr>
<tr>
<td>100</td>
<td>5.861(0.006)</td>
<td>5.856(0.006)</td>
<td>5.845(0.006)</td>
<td>5.821(0.006)</td>
<td>5.806</td>
</tr>
<tr>
<td>150</td>
<td>7.618(0.010)</td>
<td>7.612(0.009)</td>
<td>7.584(0.010)</td>
<td>7.542(0.010)</td>
<td>7.509</td>
</tr>
</tbody>
</table>

### $K = 100$ (ATM)

<table>
<thead>
<tr>
<th>$T$</th>
<th>$U_0$</th>
<th>$w_L^3$</th>
<th>$w_{La}^3$</th>
<th>$w_{Lb}^3$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>7.612(0.004)</td>
<td>7.608(0.004)</td>
<td>7.596(0.004)</td>
<td>7.581(0.004)</td>
<td>7.579</td>
</tr>
<tr>
<td>100</td>
<td>10.334(0.009)</td>
<td>10.327(0.008)</td>
<td>10.299(0.009)</td>
<td>10.254(0.009)</td>
<td>10.223</td>
</tr>
<tr>
<td>150</td>
<td>12.274(0.015)</td>
<td>12.268(0.013)</td>
<td>12.225(0.014)</td>
<td>12.123(0.014)</td>
<td>12.064</td>
</tr>
</tbody>
</table>

### $K = 110$ (ITM)

<table>
<thead>
<tr>
<th>$T$</th>
<th>$U_0$</th>
<th>$w_L^3$</th>
<th>$w_{La}^3$</th>
<th>$w_{Lb}^3$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>13.704(0.006)</td>
<td>13.696(0.006)</td>
<td>13.671(0.006)</td>
<td>13.629(0.006)</td>
<td>13.616</td>
</tr>
<tr>
<td>100</td>
<td>16.253(0.013)</td>
<td>16.241(0.011)</td>
<td>16.195(0.012)</td>
<td>16.089(0.012)</td>
<td>16.037</td>
</tr>
<tr>
<td>150</td>
<td>18.151(0.019)</td>
<td>18.145(0.016)</td>
<td>18.066(0.018)</td>
<td>17.888(0.019)</td>
<td>17.782</td>
</tr>
</tbody>
</table>

1. $U_0 > w_L^3 > w_{La}^3 > w_{Lb}^3 > V_0$. \( \cdots \) \( L \leq L^a \leq L^b \), Lower Bound Effect

2. $w_{Lb}^3 > V_0$. The proposed methods can improve the upper bound efficiently but cannot attain the true price.
Bermudan Max Call Option on five Assets

- Suppose that the price processes $S^i$ for $i = 1, \ldots, 5$ are given by $S^i_0 = S_0$, 
  $$S^i_k = S^i_{k-1} \exp \left( \left( -q - \frac{\sigma^2}{2} \right) \Delta t + \sigma \sqrt{\Delta t} \xi^i_k \right), \quad k = 1, \ldots, T.$$ 

- $H_k = \max \left( \max_{1 \leq i \leq 5} S^i_k - Ke^{-r \Delta t}, 0 \right), \quad k = 0, 1, \ldots, T.$
- $K = 100$, $q = 0.1$, $\sigma = 0.2$, $r = 0.05$, $T = \frac{3}{\Delta t}$.
- The number of paths for calculating the expectation and the conditional expectation are 250,000 and 500 respectively.
- An upper bound process is generated by the single European options.
- A lower bound process is based on the least square method.
- The true price $V_0$ is the point estimate in Broadie and Glasserman (2004).

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$S_0$</th>
<th>$U_0$</th>
<th>$w^1_L$</th>
<th>$w^2_L$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>90</td>
<td>17.572 (0.015)</td>
<td>16.866 (0.015)</td>
<td>16.496 (0.014)</td>
<td>16.474</td>
</tr>
<tr>
<td>1/2</td>
<td>100</td>
<td>28.038 (0.019)</td>
<td>26.645 (0.020)</td>
<td>25.997 (0.019)</td>
<td>25.920</td>
</tr>
<tr>
<td>1/2</td>
<td>110</td>
<td>39.721 (0.023)</td>
<td>37.545 (0.024)</td>
<td>36.615 (0.023)</td>
<td>36.497</td>
</tr>
<tr>
<td>1/3</td>
<td>90</td>
<td>17.804 (0.014)</td>
<td>17.033 (0.014)</td>
<td>16.677 (0.013)</td>
<td>16.659</td>
</tr>
<tr>
<td>1/3</td>
<td>100</td>
<td>28.296 (0.018)</td>
<td>26.855 (0.018)</td>
<td>26.264 (0.017)</td>
<td>26.158</td>
</tr>
<tr>
<td>1/3</td>
<td>110</td>
<td>39.956 (0.021)</td>
<td>37.816 (0.022)</td>
<td>36.994 (0.021)</td>
<td>36.782</td>
</tr>
</tbody>
</table>
We have proposed a simple and computationally tractable improvement method for the upper bound of American options.

- The method is based on two stopping times. The stopping times are generated from a lower bound process of the continuation value.
- A better, namely higher lower bound process gives a greater improvement of the upper bound.
- Our method can be used together with the approximation of lower bound process by the least square method.
References


