Pricing and Hedging American Options under Exponential Subordinated Levy Processes by Malliavin Calculus

BFS2010 Presentation, 6/22-6/26, 2010, Toronto, Canada

Yongzeng Lai (ylai@wlu.ca) & Yiqi Wang (yiqi.wang@ca.pwc.com)

Wilfrid Laurier University, Waterloo, Ontario, Canada & Pricewaterhouse Coopers LLP

June 23, 2010
Outline

- Introduction
Outline

- Introduction
- Single-asset case
Outline

- Introduction
- Single-asset case
- Multi-asset case
Outline

- Introduction
- Single-asset case
- Multi-asset case
- Algorithms
Outline

- Introduction
- Single-asset case
- Multi-asset case
- Algorithms
- Questions
High dimensional American options pricing and sensitivities estimation are still challenging even under the Black-Scholes-Merton’s (BSM) model.
High dimensional American options pricing and sensitivities estimation are still challenging even under the Black-Scholes-Merton’s (BSM) model.

Two main methods under the BSM model
High dimensional American options pricing and sensitivities estimation are still challenging even under the Black-Scholes-Merton’s (BSM) model.

Two main methods under the BSM model

- Regression based method and its variations, e.g., Longstaff-Schwartz’s least-squares method, etc.
High dimensional American options pricing and sensitivities estimation are still challenging even under the Black-Scholes-Merton’s (BSM) model.

Two main methods under the BSM model

1. Regression based method and its variations, e.g., Longstaff-Schwartz’s least-squares method, etc.
2. Bally et al.’s Malliavin calculus method.
High dimensional American options pricing and sensitivities estimation are still challenging even under the Black-Scholes-Merton’s (BSM) model.

Two main methods under the BSM model

1. Regression based method and its variations, e.g., Longstaff-Schwartz’s least-squares method, etc.
2. Bally et al.’s Malliavin calculus method.
   - The main idea of this method is to express a conditional expectation as the ratio of two unconditional expectations.
We discuss the pricing and sensitivity estimation of American options under a special type of Levy processes - subordinated Levy processes.
We discuss the pricing and sensitivity estimation of American options under a special type of Levy processes - subordinated Levy processes.

A subordinated Levy process (SLP) is also called subordinated Brownian motion (SBM) or time changed Brownian motion.
We discuss the pricing and sensitivity estimation of American options under a special type of Levy processes - subordinated Levy processes.

A subordinated Levy process (SLP) is also called subordinated Brownian motion (SBM) or time changed Brownian motion.

Two typical such processes are normal inverse Gaussian (NIG) process and variance gamma (VG) process.
Empirical studies show that some Levy processes, e.g., generalized hyperbolic (GH) processes, can fit real financial data much better than the (geometric) Brownian motions.
Empirical studies show that some Levy processes, e.g., generalized hyperbolic (GH) processes, can fit real financial data much better than the (geometric) Brownian motions.

Both NIG & VG processes are special cases of GH processes.
**Figure:** Comparisons of densities for Google data set

Densities: Google

- Normal
- NIG
- Empirical
Figure: Comparisons of densities for IBM data set
Figure: Comparisons of densities for RIM data set
Figure: Comparisons of densities for SINOPEC data set
Figure: Comparisons of densities for BANKOFCHINA data set
Figure: Comparisons of densities for MAOTAI data set
The integration by parts (IBP) property is valid for two square integrable r.v.s $F$ & $G$, denoted by $IBP(F, G)$,
The integration by parts (IBP) property is valid for two square integrable r.v.s $F$ & $G$, denoted by $IBP(F, G)$, if there exists a square integrable random weight $\pi_F(G)$ such that

$$E(\phi'(F)G) = E(\phi(F)\pi_F(G))$$
The integration by parts (IBP) property is valid for two square integrable r.v.s \( F \) & \( G \), denoted by \( IBP(F,G) \), if \( \exists \) a square integrable random weight \( \pi_F(G) \) such that

\[
E(\phi'(F)G) = E(\phi(F)\pi_F(G))
\]

for any \( \phi \in C^\infty_b(\mathbb{R}) \), the set of bounded and infinitely differentiable functions.
Lemma 1 (Bally et al.): If both $IBP(F,1)$ and $IBP(F,G)$ hold, then
Lemma 1 (Bally et al.): If both IBP($F, 1$) and IBP($F, G$) hold, then

$$E(G \mid F = \alpha) = \frac{E[H(F - \alpha)\pi_F(G)]}{E[H(F - \alpha)\pi_F(1)]}$$
Lemma 1 (Bally et al.): If both $IBP(F, 1)$ and $IBP(F, G)$ hold, then

$$E(G \mid F = \alpha) = \frac{E[H(F - \alpha)\pi_F(G)]}{E[H(F - \alpha)\pi_F(1)]}$$

with the convention that $E(G \mid F = \alpha) = 0$ if $E[H(F - \alpha)\pi_F(1)] = 0$;
Lemma 1 (Bally et al.): If both IBP\((F, 1)\) and IBP\((F, G)\) hold, then
\[
E(G \mid F = \alpha) = \frac{E[H(F - \alpha)\pi_F(G)]}{E[H(F - \alpha)\pi_F(1)]}
\]
with the convention that \(E(G \mid F = \alpha) = 0\) if
\[
E[H(F - \alpha)\pi_F(1)] = 0;
\]
\(H(x) = 1_{\{x \geq 0\}}(x), \ x \in \mathbb{R}\) - the Heaviside function.
**Lemma 2** (Bally *et al.*): If \( X = x \exp(\mu + \sigma \Delta) \) with \( \Delta \sim \mathcal{N}(0, \delta) \), then
Lemma 2 (Bally et al.): If $X = x \exp(\mu + \sigma \Delta)$ with $\Delta \sim N(0, \delta)$, then

$$E \left[ f'(X) g(X) \right] = E \left\{ f(X) \left[ \frac{g(X)}{\sigma X} \left( \frac{\Delta}{\delta} + \sigma \right) - g'(X) \right] \right\}$$

for $f, g \in C^1$. 

---

Yongzeng Lai (ylai@wlu.ca) & Yiqi Wang (yiqi@wlu.ca) (Wilfrid Laurier University, Waterloo, Ontario, Canada & PricewaterhouseCoopers)
Consider an asset whose price process is given by the following exponential subordinated Levy process.
Consider an asset whose price process is given by the following exponential subordinated Levy process

\[ S_t = S_0 \exp (\mu Y_t + \sigma W_{Y_t}), \quad t > 0. \]
Consider an asset whose price process is given by the following exponential subordinated Levy process

\[ S_t = S_0 \exp(\mu Y_t + \sigma W_{Y_t}), \quad t > 0. \]

where \( \{ Y_t \} \) is a subordinator process.
Consider an asset whose price process is given by the following exponential subordinated Levy process

$$S_t = S_0 \exp (\mu Y_t + \sigma W_{Y_t}), \quad t > 0.$$ 

where \(\{Y_t\}\) is a subordinator process.

Denote \(\mathcal{F}_t = \sigma(Y_r, r \in [0, t])\), the \(\sigma\) field generated by \(\{Y_r, r \in [0, t]\}\).
Proposition 1: Assume that $S_t$ is given as before. Let $0 < s < t$, $g : \mathbb{R} \to \mathbb{R}$ be a function with polynomial growth.
Proposition 1: Assume that $S_t$ is given as before. Let $0 < s < t$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function with polynomial growth.

(1) The $IBP(S_s, g(S_t))$ property holds, i.e.,
Proposition 1: Assume that $S_t$ is given as before. Let $0 < s < t$, $g : \mathbb{R} \to \mathbb{R}$ be a function with polynomial growth.

(1) The IBP$(S_s, g(S_t))$ property holds, i.e.,

$$E \left[ \phi'(S_s)g(S_t) \right] = E \left\{ \phi(S_s)\pi_s[g](S_t) \right\}, \quad \forall \phi \in C_b^\infty(\mathbb{R})$$
Proposition 1: Assume that $S_t$ is given as before. Let $0 < s < t$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function with polynomial growth.

(1) The $IBP(S_s, g(S_t))$ property holds, i.e.,

$$E \left[ \phi'(S_s) g(S_t) \right] = E \left\{ \phi(S_s) \pi_s[g](S_t) \right\} , \ ∀ \phi \in C^\infty_b(\mathbb{R})$$

where

$$\pi_s[g](S_t) = \frac{g(S_t) \Delta W_{s,t}}{\sigma Y_s (Y_t - Y_s)}$$
Proposition 1: Assume that $S_t$ is given as before. Let $0 < s < t$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function with polynomial growth.

(1) The $IBP(S_s, g(S_t))$ property holds, i.e.,

$$E \left[ \phi'(S_s) g(S_t) \right] = E \left\{ \phi(S_s) \pi_s[g](S_t) \right\}, \forall \phi \in C^\infty_b(\mathbb{R})$$

where

$$\pi_s[g](S_t) = \frac{g(S_t) \Delta W_{s,t}}{\sigma Y_s(Y_t - Y_s)}$$

with $\Delta W_{s,t} = (Y_t - Y_s) W_{Y_s} - Y_s (W_{Y_t} - W_{Y_s}) + \sigma Y_s (Y_t - Y_s) = Y_t W_{Y_s} - Y_s W_{Y_t} + \sigma Y_s (Y_t - Y_s)$. 
(2) For fixed $\alpha > 0$, let $\phi \in C^1_b(\mathbb{R}) = \{\phi \in C^1(\mathbb{R}) \text{ and } \phi \text{ bounded}\}$ be such that
(2) For fixed $\alpha > 0$, let $\phi \in C^1_b(\mathbb{R}) = \{\phi \in C^1(\mathbb{R})$ and $\phi$ bounded$\}$ be such that

$\text{supp}(\phi') \subset B_\varepsilon(\alpha) = (\alpha - \varepsilon, \alpha + \varepsilon)$ with $\varepsilon > 0$. 
(2) For fixed \( \alpha > 0 \), let \( \phi \in C_b^1(\mathbb{R}) = \{ \phi \in C^1(\mathbb{R}) \text{ and } \phi \text{ bounded} \} \) be such that

\[ \text{supp}(\phi') \subset B_\varepsilon(\alpha) = (\alpha - \varepsilon, \alpha + \varepsilon) \text{ with } \varepsilon > 0. \]

Let \( \tilde{\psi} \in C_b^1(\mathbb{R}) \) be such that \( \tilde{\psi} |_{B_\varepsilon(\alpha)} = 1 \). Then,
(2) For fixed $\alpha > 0$, let $\phi \in C^1_b(\mathbb{R}) = \{\phi \in C^1(\mathbb{R}) \text{ and } \phi \text{ bounded}\}$ be such that $\text{supp}(\phi') \subset B_\varepsilon(\alpha) = (\alpha - \varepsilon, \alpha + \varepsilon)$ with $\varepsilon > 0$.

Let $\tilde{\psi} \in C^1_b(\mathbb{R})$ be such that $\tilde{\psi} \big|_{B_\varepsilon(\alpha)} = 1$. Then,

$$E \left[ \phi'(S_s)g(S_t) \right] = E \left\{ \phi(S_s)\pi_s\tilde{\psi}[g](S_t) \right\}, \quad \forall \phi \in C^\infty_b(\mathbb{R})$$
Single-asset case (3)

(2) For fixed $\alpha > 0$, let $\phi \in C^1_b(\mathbb{R}) = \{ \phi \in C^1(\mathbb{R}) \text{ and } \phi \text{ bounded} \}$ be such that

$supp(\phi') \subset B_\varepsilon(\alpha) = (\alpha - \varepsilon, \alpha + \varepsilon)$ with $\varepsilon > 0$.

Let $\tilde{\psi} \in C^1_b(\mathbb{R})$ be such that $\tilde{\psi} \big|_{B_\varepsilon(\alpha)} = 1$. Then,

$$E \left[ \phi'(S_s)g(S_t) \right] = E \left\{ \phi(S_s)\pi_{\tilde{\psi}}[g](S_t) \right\}, \text{ } \forall \phi \in C^\infty_b(\mathbb{R})$$

where

$$\pi_{\tilde{\psi}}[g](S_t) = \frac{g(S_t)\Delta_{\tilde{\psi}}W_{s,t}}{\sigma Y_s(Y_t - Y_s)}$$
(2) For fixed $\alpha > 0$, let $\phi \in C^1_b(\mathbb{R}) = \{ \phi \in C^1(\mathbb{R}) \text{ and } \phi \text{ bounded} \}$ be such that 

$\text{supp}(\phi') \subset B_\varepsilon(\alpha) = (\alpha - \varepsilon, \alpha + \varepsilon)$ with $\varepsilon > 0$.

Let $\tilde{\psi} \in C^1_b(\mathbb{R})$ be such that $\tilde{\psi} \big|_{B_\varepsilon(\alpha)} = 1$. Then,

$$E \left[ \phi'(S_s)g(S_t) \right] = E \left\{ \phi(S_s) \pi_s \tilde{\psi}[g](S_t) \right\}, \quad \forall \phi \in C^\infty_b(\mathbb{R})$$

where

$$\pi_s \tilde{\psi}[g](S_t) = \frac{g(S_t) \Delta \tilde{\psi} W_{s,t}}{\sigma Y_s (Y_t - Y_s)}$$

with $\Delta \tilde{\psi} W_{s,t} = \tilde{\psi}(S_s) \Delta W_{s,t} - \tilde{\psi}'(S_s) S_s \sigma Y_s (Y_t - Y_s)$. 

---

Yongzeng Lai (ylai@wlu.ca) & Yiqi Wang (yiqi.wang@ca.pwc.com) (Wilfrid Laurier University, Waterloo, Ontario, Canada & Pricewaterhouse)
(3) The IBP \(S_s, \frac{g(S_t) \Delta W_{s,t}}{\sigma Y_s (Y_t - Y_s)}\) property holds, i.e.,
(3) The \( IBP(S_s, \frac{g(S_t)\Delta W_{s,t}}{\sigma Y_s(Y_t - Y_s)} ) \) property holds, i.e.,

\[
E \left[ \phi'(S_s) \frac{g(S_t)\Delta W_{s,t}}{\sigma Y_s(Y_t - Y_s)S_s} \right] = E \left\{ \phi(S_s)\pi_s[g](S_t) \right\}, \quad \forall \phi \in C^\infty_b(\mathbb{R})
\]
(3) The \( IBP(S_s, \frac{g(S_t)\Delta W_{s,t}}{\sigma Y_s(Y_t - Y_s)} ) \) property holds, i.e.,

\[
E \left[ \phi'(S_s) \frac{g(S_t)\Delta W_{s,t}}{\sigma Y_s(Y_t - Y_s)} S_s \right] = E \left\{ \phi(S_s) \overline{\pi}_s[g](S_t) \right\}, \quad \forall \phi \in C_b^\infty(\mathbb{R})
\]

where

\[
\overline{\pi}_s[g](S_t) = \frac{g(S_t)}{\sigma Y_s(Y_t - Y_s)} S_s^2 \left[ \frac{(\Delta W_{s,t})^2}{\sigma Y_s(Y_t - Y_s)} + \Delta W_{s,t} - \frac{Y_t}{\sigma} \right].
\]
Theorem 1. (Conditional expectation formula without localization)
Theorem 1. (Conditional expectation formula without localization)

(1) For any $0 < s < t$, $\alpha > 0$ and $\Phi$,

$$E \left[ \Phi(S_t) \mid S_s = \alpha \right] = \frac{T_{s,t}[\Phi](\alpha)}{T_{s,t}[1](\alpha)},$$


**Theorem 1.** (Conditional expectation formula without localization)

(1) For any $0 < s < t$, $\alpha > 0$ and $\Phi$,

$$E \left[ \Phi(S_t) | S_s = \alpha \right] = \frac{T_{s,t}[\Phi](\alpha)}{T_{s,t}[1](\alpha)},$$

where

$$T_{s,t}[f](\alpha) = E \left[ f(S_t) \frac{H(S_s - \alpha)}{\sigma Y_s (Y_t - Y_s) S_s} \Delta W_{s,t} \right]$$
(2) For any $0 < s < t$, $\alpha > 0$ and $\Phi$,

$$\partial_\alpha E \left[ \Phi(S_t) | S_s = \alpha \right] = \frac{R_{s,t}[\Phi](\alpha) T_{s,t}[1](\alpha) - R_{s,t}[1](\alpha) T_{s,t}[\Phi](\alpha)}{(T_{s,t}[1](\alpha))^2},$$
(2) For any $0 < s < t$, $\alpha > 0$ and $\Phi$,

$$
\partial_\alpha E \left[ \Phi(S_t) \left| S_s = \alpha \right. \right] = \frac{R_{s,t}[\Phi](\alpha) T_{s,t}[1](\alpha) - R_{s,t}[1](\alpha) T_{s,t}[\Phi](\alpha)}{(T_{s,t}[1](\alpha))^2},
$$

where

$$R_{s,t}[f](\alpha) =
- E \left[ f(S_t) \frac{H(S_s - \alpha)}{\sigma Y_s(Y_t - Y_s) S_s^2} \left( \frac{\Delta W_{s,t}^2}{\sigma Y_s(Y_t - Y_s)} + \Delta W_{s,t} - \frac{Y_t}{\sigma} \right) \right].$$
Lemma 3 (Localization)
Lemma 3 (Localization)

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}_+ = [0, \infty)$ be a PDF and $\Psi$ be its corresponding CDF.
Lemma 3 (Localization)

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}_+ = [0, \infty)$ be a PDF and $\Psi$ be its corresponding CDF. Then, conditional expressions are also true if $T_{s,t}[f](\alpha)$ is replaced by
Lemma 3 (Localization)

Let $\psi : \mathbb{R} \to \mathbb{R}_{+} = [0, \infty)$ be a PDF and $\Psi$ be its corresponding CDF. Then, conditional expressions are also true if $\mathbb{T}_{s,t}[f](\alpha)$ is replaced by

$$\mathbb{T}_{s,t}^{\psi}[f](\alpha) = \mathbb{E}\left[f(S_t) \left(\psi(S_s - \alpha) + \frac{H(S_s - \alpha) - \Psi(S_s - \alpha)}{\sigma Y_s(Y_t - Y_s)S_s} \Delta W_{s,t}\right)\right]$$
Lemma 3 (Localization)

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}_+ = [0, \infty)$ be a PDF and $\Psi$ be its corresponding CDF. Then, conditional expressions are also true if $\mathbb{T}_{s,t}[f](\alpha)$ is replaced by

$$
\mathbb{T}_{s,t}^\psi[f](\alpha) = E \left[ f(S_t) \left( \psi(S_s - \alpha) + \frac{H(S_s - \alpha) - \Psi(S_s - \alpha)}{\sigma Y_s (Y_t - Y_s) S_s} \Delta W_{s,t} \right) \right]
$$

and $\mathbb{R}_{s,t}[f](\alpha)$ is replaced by

$$
\mathbb{R}_{s,t}^\psi[f](\alpha) = -E \left\{ f(S_t) \left[ \psi(S_s - \alpha) \frac{\Delta W_{s,t}}{\sigma Y_s (Y_t - Y_s)} + \frac{H(S_s - \alpha) - \Psi(S_s - \alpha)}{\sigma Y_s (Y_t - Y_s) S_s} \left( \frac{\Delta W_{s,t}^2}{\sigma Y_s (Y_t - Y_s)} + \Delta W_{s,t} - \frac{Y_t}{\sigma} \right) \right] \right\},
$$

respectively.
Theorem 2: (Conditional expectation formula with localization)
Theorem 2: (Conditional expectation formula with localization)

Let $\psi : \mathbb{R} \to [0,\infty)$ be a PDF and $\Psi$ be its corresponding CDF. For any $0 < s < t$, $\alpha > 0$ and $\Phi$, we have
Theorem 2: (Conditional expectation formula with localization)

Let $\psi : \mathbb{R} \to [0, \infty)$ be a PDF and $\Psi$ be its corresponding CDF. For any $0 < s < t$, $\alpha > 0$ and $\Phi$, we have

$$E [\Phi(S_t) | S_s = \alpha] = \frac{T^\psi_{s,t} [\Phi](\alpha)}{T^\psi_{s,t} [1](\alpha)},$$
Theorem 2: (Conditional expectation formula with localization)

Let $\psi : \mathbb{R} \to [0,\infty)$ be a PDF and $\Psi$ be its corresponding CDF. For any $0 < s < t$, $\alpha > 0$ and $\Phi$, we have

$$E[\Phi(S_t)|S_s = \alpha] = \frac{T_{s,t}^{\psi}[\Phi](\alpha)}{T_{s,t}^{\psi}[1](\alpha)},$$

and

$$\partial_\alpha E[\Phi(S_t)|S_s = \alpha] = \frac{R_{s,t}^{\psi}[\Phi](\alpha)T_{s,t}^{\psi}[1](\alpha) - R_{s,t}^{\psi}[1](\alpha)T_{s,t}^{\psi}[\Phi](\alpha)}{\left(T_{s,t}^{\psi}[1](\alpha)\right)^2},$$
Single-asset case (8)

- **Theorem 2**: (Conditional expectation formula with localization)
- Let $\psi : \mathbb{R} \rightarrow [0, \infty)$ be a PDF and $\Psi$ be its corresponding CDF. For any $0 < s < t$, $\alpha > 0$ and $\Phi$, we have

$$E \left[ \Phi(S_t) \big| S_s = \alpha \right] = \frac{T_{s,t}^{\psi}[\Phi](\alpha)}{T_{s,t}^{\psi}[1](\alpha)},$$

and

$$\partial_\alpha E \left[ \Phi(S_t) \big| S_s = \alpha \right] = \frac{R_{s,t}^{\psi}[\Phi](\alpha)T_{s,t}^{\psi}[1](\alpha) - R_{s,t}^{\psi}[1](\alpha)T_{s,t}^{\psi}[\Phi](\alpha)}{\left(T_{s,t}^{\psi}[1](\alpha)\right)^2},$$

where $T_{s,t}^{\psi}[\Phi](\alpha)$ and $R_{s,t}^{\psi}[\Phi](\alpha)$ are given in the above.
Consider a market model with $d$ assets whose price processes are given by the following exponential subordinated Levy processes.
Consider a market model with $d$ assets whose price processes are given by the following exponential subordinated Levy processes

$$S_{i;t} = S_{i;0} \exp \left( \mu_i Y_t + \sum_{l=1}^{d} c_{il} W_{l;Y_t} \right), \quad i = 1, \ldots, d,$$
Multi-asset case (1)

Consider a market model with $d$ assets whose price processes are given by the following exponential subordinated Levy processes

$$S_{i; t} = S_{i; 0} \exp \left( \mu_i Y_t + \sum_{l=1}^{d} c_{il} W_l; Y_t \right), \quad i = 1, \ldots, d,$$

where $C = (c_{ij})_{d \times d}$ is a matrix such that $\sigma = (\sigma_{ij})_{d \times d} = CC'$ is the covariance matrix (e.g., $C$ can be taken as the Cholesky decomposition of $\sigma$).
Multi-asset case (1)

Consider a market model with $d$ assets whose price processes are given by the following exponential subordinated Levy processes

$$S_{i,t} = S_{i,0} \exp \left( \mu_i Y_t + \sum_{l=1}^{d} c_{il} W_{l;Y_t} \right), \ i = 1, \cdots, d,$$

where $C = (c_{ij})_{d \times d}$ is a matrix such that $\sigma = (\sigma_{ij})_{d \times d} = CC'$ is the covariance matrix (e.g., $C$ can be taken as the Cholesky decomposition of $\sigma$).

$\{Y_t\}$ is a subordinator process. For simplicity, assume that $C$ is lower triangular, i.e., $c_{ij} = 0$ for $i < j$. 
Multi-asset case (1)

Consider a market model with $d$ assets whose price processes are given by the following exponential subordinated Levy processes

$$S_{i;t} = S_{i;0} \exp \left( \mu_i Y_t + \sum_{l=1}^{d} c_{il} W_{l;Y_t} \right), \ i = 1, \cdots, d,$$

where $C = (c_{ij})_{d \times d}$ is a matrix such that $\sigma = (\sigma_{ij})_{d \times d} = CC'$ is the covariance matrix (e.g., $C$ can be taken as the Cholesky decomposition of $\sigma$),

$\{Y_t\}$ is a subordinator process. For simplicity, assume that $C$ is lower triangular, i.e., $c_{ij} = 0$ for $i < j$.

Thus,

$$S_{i;t} = S_{i;0} \exp \left( \mu_i Y_t + \sum_{l=1}^{i} c_{il} W_{l;Y_t} \right), \ i = 1, \cdots, d,$$
Denote $\mathcal{F}_t = \sigma(Y_r, r \in [0, t])$. 
Multi-asset case (2)

- Denote $\mathcal{F}_t = \sigma(Y_r, r \in [0, t])$.
- Let $0 < s < t$, $\alpha > 0$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$ - the set of measurable functions with polynomial growth.
Multi-asset case (2)

- Denote $\mathcal{F}_t = \sigma(Y_r, r \in [0, t])$.
- Let $0 < s < t$, $\alpha > 0$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$ - the set of measurable functions with polynomial growth.
- To express the conditional expectation $E[\Phi(S_t)|S_s = \alpha]$, we try to use the results in one-dimensional case.
Denote $\mathcal{F}_t = \sigma(Y_r, r \in [0, t])$.

Let $0 < s < t$, $\alpha > 0$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$ - the set of measurable functions with polynomial growth.

To express the conditional expectation $E[\Phi(S_t)|S_s = \alpha]$, we try to use the results in one-dimensional case.

To this purpose, we consider an auxiliary process $\tilde{S}_t$ with independent coordinates conditional on $\mathcal{F}_t$. 
Multi-asset case (2)

- Denote $\mathcal{F}_t = \sigma(Y_r, r \in [0, t])$.
- Let $0 < s < t$, $\alpha > 0$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$ - the set of measurable functions with polynomial growth.
- To express the conditional expectation $E[\Phi(S_t) | S_s = \alpha]$, we try to use the results in one-dimensional case.
- To this purpose, we consider an auxiliary process $\tilde{S}_t$ with independent coordinates conditional on $\mathcal{F}_t$.
- Let $p_t = (p_{1,t}, \cdots, p_{d,t})$ be a fixed $C^1$ function (to be determined later) and let

$$\tilde{S}_{i;t} = S_{i;0} \exp (\mu_i Y_t + p_{it} + c_{ii} W_{i; Y_t}), \; i = 1, \cdots, d,$$
Multi-asset case (3)

- **Lemma 4** (Relationship between \( \{ S_t \} \) & \( \{ \tilde{S}_t \} \))
Lemma 4 (Relationship between \( \{S_t\} \) & \( \{\tilde{S}_t\} \))

Denote \( \tilde{C} = (\tilde{c}_{ij}) \), \( \tilde{c}_{ij} = \frac{c_{ij}}{c_{jj}} \), \( i, j = 1, \ldots, d \).
Lemma 4 (Relationship between $\{S_t\} \& \{\tilde{S}_t\}$)

Denote $\tilde{C} = (\tilde{c}_{ij})$, $\tilde{c}_{ij} = \frac{c_{ij}}{c_{jj}}$, $i, j = 1, \ldots, d$.

If $\hat{C} = \tilde{C}^{-1}$ exists, then any $t \geq 0$,

$$S_t = F_t(\tilde{S}_t) \text{ and } \tilde{S}_t = G_t(S_t) = F_t^{-1}(S_t)$$
Lemma 4 (Relationship between $\{S_t\}$ & $\{\tilde{S}_t\}$)
Denote $\tilde{C} = (\tilde{c}_{ij})$, $\tilde{c}_{ij} = \frac{c_{ij}}{c_{jj}}$, $i, j = 1, \cdots, d$.

If $\hat{C} = \tilde{C}^{-1}$ exists, then any $t \geq 0$,

$$S_t = F_t(\tilde{S}_t) \text{ and } \tilde{S}_t = G_t(S_t) = F_t^{-1}(S_t)$$

where $F_t, G_t : \mathbb{R}^d_+ \rightarrow \mathbb{R}^d_+$ are given by

$$\ln F_t(y) = -\tilde{C}p_t + \tilde{C} \ln y + (I - \tilde{C})(\ln S_0 + \mu Y_t),$$
Multi-asset case (3)

Lemma 4 (Relationship between \(\{S_t\} & \{\tilde{S}_t\}\))

Denote \(\tilde{C} = (\tilde{c}_{ij})\), \(\tilde{c}_{ij} = \frac{c_{ij}}{c_{jj}}\), \(i, j = 1, \cdots, d\).

If \(\hat{C} = \tilde{C}^{-1}\) exists, then any \(t \geq 0\),

\[
S_t = F_t(\tilde{S}_t) \quad \text{and} \quad \tilde{S}_t = G_t(S_t) = F_t^{-1}(S_t)
\]

where \(F_t, G_t : \mathbb{R}_+^d \to \mathbb{R}_+^d\) are given by

\[
\ln F_t(y) = -\tilde{C} p_t + \tilde{C} \ln y + (I - \tilde{C})(\ln S_0 + \mu Y_t),
\]

and

\[
\ln G_t(z) = p_t + \hat{\sigma} \ln z + (I - \hat{\sigma})(\ln S_0 + \mu Y_t)
\]

respectively,
Multi-asset case (3)

- **Lemma 4** (Relationship between \{S_t\} & \{\tilde{S}_t\})
- Denote \(\tilde{C} = (\tilde{c}_{ij}), \tilde{c}_{ij} = \frac{c_{ij}}{c_{jj}}, i, j = 1, \cdots, d\).
- If \(\hat{C} = \tilde{C}^{-1}\) exists, then any \(t \geq 0\),
  \[
  S_t = F_t(\tilde{S}_t) \text{ and } \tilde{S}_t = G_t(S_t) = F_{t^{-1}}(S_t)
  \]
- where \(F_t, G_t : \mathbb{R}_d^+ \rightarrow \mathbb{R}_d^+\) are given by
  \[
  \ln F_t(y) = -\tilde{C} p_t + \tilde{C} \ln y + (I - \tilde{C})(\ln S_0 + \mu Y_t),
  \]
  and
  \[
  \ln G_t(z) = p_t + \hat{\sigma} \ln z + (I - \hat{\sigma})(\ln S_0 + \mu Y_t)
  \]
  respectively,
- with \(y = (y_1, \cdots, y_d), z = (z_1, \cdots, z_d) \in \mathbb{R}_d^+ = \{u \in \mathbb{R}^d, u_i > 0, i = 1, \cdots, d\}\),
**Lemma 4** (Relationship between \( \{ S_t \} \) & \( \{ \tilde{S}_t \} \))

Denote \( \tilde{C} = (\tilde{c}_{ij}) \), \( \tilde{c}_{ij} = \frac{c_{ij}}{c_{jj}} \), \( i, j = 1, \ldots, d \).

If \( \hat{C} = \tilde{C}^{-1} \) exists, then any \( t \geq 0 \),

\[
S_t = F_t(\tilde{S}_t) \quad \text{and} \quad \tilde{S}_t = G_t(S_t) = F_t^{-1}(S_t)
\]

where \( F_t, G_t : \mathbb{R}^d_+ \rightarrow \mathbb{R}^d_+ \) are given by

\[
\ln F_t(y) = -\tilde{C} p_t + \tilde{C} \ln y + (I - \tilde{C})(\ln S_0 + \mu Y_t),
\]

and

\[
\ln G_t(z) = p_t + \hat{\sigma} \ln z + (I - \hat{\sigma})(\ln S_0 + \mu Y_t)
\]

respectively,

with \( y = (y_1, \cdots, y_d) \), \( z = (z_1, \cdots, z_d) \) \( \in \mathbb{R}^d_+ = \{ u \in \mathbb{R}^d, u_i > 0, i = 1, \cdots, d \} \),

and \( \ln u = (\ln u_1, \cdots, \ln u_d) \) if \( y_i > 0 \) for \( i = 1, \cdots, d \).
Lemma 4 ⇒
Multi-asset case (4)

Lemma 4 ⇒

\[ S_{i; t} = F_{i; t}(\tilde{S}_t) = S_{i; 0} e^{\mu_i Y_t} \prod_{l=1}^{d} \left( \frac{\tilde{S}_{l; t}}{\tilde{S}_{l; 0}} e^{-\left( \mu_l + \sum_{l=1}^{d} \tilde{c}_{il} p_{l; t} \right) Y_t} \right)^{\tilde{c}_{il}}, \quad i = 1, \ldots, d, \]
Multi-asset case (4)

Lemma 4 \[ \Rightarrow \]

\[ S_{i;t} = F_{i;t}(\tilde{S}_t) = S_{i;0} e^{\mu_i Y_t} \prod_{l=1}^{d} \left( \frac{\tilde{S}_{l;t}}{S_{l;0}} e^{-\mu_l \sum_{l=1}^{d} \tilde{c}_{il} p_{l;t}} Y_t \right)^{\tilde{c}_{il}}, \quad i = 1, \ldots, d, \]

\[ \tilde{S}_{i;t} = G_{i;t}(S_t) = S_{i;0} e^{p_{i;t}} \prod_{l=1}^{d} \left( \frac{S_{l;t}}{S_{l;0}} e^{-\mu_l Y_t} \right)^{\tilde{c}_{il}}, \quad i = 1, \ldots, d. \]
Multi-asset case (4)

- **Theorem 3**: (Conditional expectation formula without localization)
Multi-asset case (4)

- **Theorem 3**: (Conditional expectation formula without localization)
- (1) For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, we have

$$E[\Phi(S_t) | S_s = \alpha] = \frac{\mathbb{T}_{s,t}[\Phi](\alpha)}{\mathbb{T}_{s,t}[1](\alpha)},$$
Theorem 3: (Conditional expectation formula without localization)

(1) For any $0 < s < t$, $\alpha \in \mathbb{R}_+^d$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, we have

$$E[\Phi(S_t)|S_s = \alpha] = \frac{T_{s,t}[\Phi](\alpha)}{T_{s,t}[1](\alpha)},$$

where

$$T_{s,t}[f](\alpha) = E \left[ f(S_t) \prod_{l=1}^{d} \frac{H(\tilde{S}_l; s - \tilde{\alpha}_l)}{c_{ll} Y_s (Y_t - Y_s) \tilde{S}_l; s} \Delta W_{s,t;l} \right]$$

Yongzeng Lai (ylai@wlu.ca) & Yiqi Wang (yiqi.wang@ca.pwc.com) (Wilfrid Laurier University, Waterloo, Ontario, Canada & PricewaterhouseCoopers Information and Technology Services Canada Inc., Toronto, Ontario, Canada)
Theorem 3: (Conditional expectation formula without localization)

(1) For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, we have

$$E[\Phi(S_t)|S_s = \alpha] = \frac{T_{s,t}[\Phi](\alpha)}{T_{s,t}[1](\alpha)},$$

where

$$T_{s,t}[f](\alpha) = E\left[f(S_t) \prod_{l=1}^{d} \frac{H(\tilde{S}_l; s - \tilde{\alpha}_l)}{c_l Y_s(Y_t - Y_s) \tilde{S}_l; s} \Delta W_{s,t;l}\right]$$

with $\tilde{S}_s = G_s(S_s)$, $\tilde{\alpha} = G_s(\alpha)$, and $H(\alpha)$ the same as before,
Multi-asset case (4)

**Theorem 3**: (Conditional expectation formula without localization)

(1) For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$ and $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, we have

$$E[\Phi(S_t)|S_s = \alpha] = \frac{T_{s,t}[\Phi](\alpha)}{T_{s,t}[1](\alpha)},$$

where

$$T_{s,t}[f](\alpha) = E \left[ f(S_t) \prod_{l=1}^{d} \frac{H(\tilde{S}_{l; s} - \tilde{\alpha}_l)}{c_{ll} Y_s(Y_t - Y_s) \tilde{S}_{l; s}} \Delta W_{s,t;l} \right]$$

with $\tilde{S}_s = G_s(S_s)$, $\tilde{\alpha} = G_s(\alpha)$, and $H(x)$ the same as before,

and $\Delta W_{s,t;l} = Y_t W_{l; Y_s} - Y_s W_{l; Y_t} + c_{ll} Y_s(Y_t - Y_s)$.
(2) For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$, $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, and $i = 1, \ldots, d$, we have

$$\partial_{\alpha_i} E \left[ \Phi(S_t) \middle| S_s = \alpha \right] = \sum_{l=1}^{d} \tilde{c}_{il} \frac{R_{s,t;l} \Phi(\alpha) T_{s,t} [1](\alpha) - R_{s,t;l} [1](\alpha) T_{s,t} \Phi(\alpha)}{(T_{s,t} [1](\alpha))^2}.$$
(2) For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$, $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, and $i = 1, \ldots, d$, we have

$$\frac{\partial}{\partial \alpha_i} \mathbb{E}[\Phi(S_t) | S_s = \alpha] = \sum_{l=1}^{d} \hat{c}_{il} \mathbb{R}_{s,t;l}[\Phi](\alpha) \mathbb{T}_{s,t}[\alpha] - \mathbb{R}_{s,t;l}[1](\alpha) \mathbb{T}_{s,t}[\Phi](\alpha) \left( \mathbb{T}_{s,t}[1](\alpha) \right)^2,$$

where $\mathbb{T}_{s,t}[f](\alpha)$ is given above and

$$\mathbb{R}_{s,t;l}[f](\alpha) = -\mathbb{E} \left\{ f(S_t) \frac{H(\tilde{S}_{l;s} - \tilde{\alpha}_l)}{c_{ll} Y_s(Y_t - Y_s) \tilde{S}_{l;s}} \left[ \frac{(\Delta W_{s,t;l})^2}{c_{ll} Y_s(Y_t - Y_s)} \right] \prod_{j=1, j \neq l}^{d} \frac{H(\tilde{S}_{j;s} - \tilde{\alpha}_j)}{c_{jj} Y_s(Y_t - Y_s) \tilde{S}_{j;s}} \Delta W_{s,t;j} \right\}.$$
Multi-asset case (6)

- **Lemma 5** (Localization)
Lemma 5 (Localization)

Let $\psi(x) = \prod_{i=1}^{d} \psi_i(x_i)$, where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, each $\psi_i : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ is a PDF with its corresponding CDF $\Psi_i$. 
Lemma 5 (Localization)

Let \( \psi(x) = \prod_{i=1}^{d} \psi_i(x_i) \), where \( x = (x_1, \cdots, x_d) \in \mathbb{R}^d \), each \( \psi_i : \mathbb{R} \to \mathbb{R}_+ = [0, \infty) \) is a PDF with its corresponding CDF \( \Psi_i \).

Then the localizations of \( T_{s,t}[f](\alpha) \) and \( R_{s,t;\ell}[f](\alpha) \) defined above, have the following forms.
**Lemma 5** (Localization)

Let \( \psi(x) = \prod_{i=1}^{d} \psi_i(x_i) \), where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), each \( \psi_i : \mathbb{R} \to \mathbb{R}_+ = [0,\infty) \) is a PDF with its corresponding CDF \( \Psi_i \).

Then the localizations of \( T_{s,t}[f](\alpha) \) and \( R_{s,t;i}[f](\alpha) \) defined above, have the following forms

\[
T_{s,t}[f](\alpha) = T_{s,t}^{\psi}[f](\alpha) = E \left[ f(S_t) \prod_{i=1}^{d} \left( \psi_i(S_{i;s} - \alpha_i) + \frac{H(\tilde{S}_{is} - \tilde{\alpha}_i) - \Psi_i(\tilde{S}_{is} - \tilde{\alpha}_i)}{c_{ii} Y_s (Y_t - Y_s) \tilde{S}_{is}} \Delta W_{s,t;i} \right) \right]
\]


Multi-asset case (7)

\[
\mathbb{R}_{s,t;l}[f](\alpha) = \mathbb{R}_{s,t;l}^\psi[f](\alpha) = -E \left\{ f(S_t) \left[ \psi_l(\tilde{S}_{l;s} - \tilde{\alpha}_l) \frac{\Delta W_{s,t;l}}{c_{ll} Y_s (Y_t - Y_s) \tilde{S}_{l;s}} + \frac{H(\tilde{S}_{l;s} - \tilde{\alpha}_l) - \Psi_l(\tilde{S}_{l;s} - \tilde{\alpha}_l)}{c_{ll} Y_s (Y_t - Y_s) (\tilde{S}_{l;s})^2} \left( \frac{\Delta W_{s,t;l}^2}{c_{ll} Y_s (Y_t - Y_s)} + \Delta W_{s,t;l} - \frac{Y_t}{c_{ll}} \right) \right] \right. \\
\left. \times \prod_{q=1, q \neq l}^d \left( \psi_q(\tilde{S}_{q;s} - \tilde{\alpha}_q) \right) + \frac{H(\tilde{S}_{q;s} - \tilde{\alpha}_q) - \Psi_q(\tilde{S}_{q;s} - \tilde{\alpha}_q)}{c_{ss} Y_s (Y_t - Y_s) \tilde{S}_{q;s}} \Delta W_{s,t;q} \right\},
\]

respectively.
Multi-asset case (8)

- **Theorem 4**: (Conditional expectation formula with localization)
Multi-asset case (8)

- **Theorem 4**: (Conditional expectation formula with localization)
  - For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$, $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, and $\psi \in \mathbb{L}_d$ (the set of $d$–dim. localization functions in product forms),
Multi-asset case (8)

- **Theorem 4**: (Conditional expectation formula with localization)
- For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$, $\Phi \in \mathcal{C}_b(\mathbb{R}^d)$, and $\psi \in \mathcal{L}_d$ (the set of $d$-dim. localization functions in product forms),
- we have

$$E[\Phi(S_t)|S_s = \alpha] = \frac{T_{s,t}^{\psi}[\Phi](\alpha)}{T_{s,t}^{\psi}[1](\alpha)},$$
Multi-asset case (8)

- **Theorem 4:** (Conditional expectation formula with localization)
  - For any $0 < s < t$, $\alpha \in \mathbb{R}_+^d$, $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, and $\psi \in \mathcal{L}_d$ (the set of $d - \text{dim. localization functions in product forms}$),
  - we have
    
    $$
    E \left[ \Phi(S_t) \mid S_s = \alpha \right] = \frac{T^\psi_{s,t}[\Phi](\alpha)}{T^\psi_{s,t}[1](\alpha)},
    $$
    
  - and for each $i = 1, \cdots, d$,
    
    $$
    \partial_{\alpha_i} E \left[ \Phi(S_t) \mid S_s = \alpha \right]
    = \sum_{l=1}^d \tilde{c}_{il} \frac{\tilde{\alpha}_l}{\alpha_i} \left( R^\psi_{s,t;l}[\Phi](\alpha) T^\psi_{s,t}[1](\alpha) - R^\psi_{s,t;l}[1](\alpha) T^\psi_{s,t}[\Phi](\alpha) \right),
    $$
    
Yongzeng Lai (ylai@wlu.ca) & Yiqi Wang (yiqi.wang@ca.pwc.com) (Wilfrid Laurier University, Waterloo, Ontario, Canada & Pricewaterhouse
**Theorem 4**: (Conditional expectation formula with localization)

For any $0 < s < t$, $\alpha \in \mathbb{R}_+^d$, $\Phi \in \mathcal{E}_b(\mathbb{R}^d)$, and $\psi \in \mathcal{L}_d$ (the set of $d - \text{dim. localization functions in product forms}$),

we have

$$E[\Phi(S_t)|S_s = \alpha] = \frac{\mathbb{T}_{s,t}[\Phi](\alpha)}{\mathbb{T}_{s,t}[1](\alpha)},$$

and for each $i = 1, \cdots, d$,

$$\partial_{\alpha_i} E[\Phi(S_t)|S_s = \alpha] = \sum_{l=1}^{d} \alpha_l \frac{\tilde{c}_{il}}{\alpha_i} \mathbb{R}_{s,t;l}[\Phi](\alpha) \mathbb{T}_{s,t}[1](\alpha) - \mathbb{R}_{s,t;l}[1](\alpha) \mathbb{R}_{s,t}[\Phi](\alpha) \left( \frac{\mathbb{T}_{s,t}[1](\alpha)}{\mathbb{T}_{s,t}[1](\alpha)} \right)^2,$$

where $\mathbb{T}_{s,t}[f](\alpha)$ and $\mathbb{R}_{s,t;l}[f](\alpha)$ are defined earlier.
The American option with payoff $\Phi$ and maturity $T$ is usually approximated by a Bermudan option with price $V(0, S_0)$ and delta $\Delta(0, S_0)$, where $S_0$ is the initial underlying asset price.
The American option with payoff \( \Phi \) and maturity \( T \) is usually approximated by a Bermudan option with price \( V(0, S_0) \) and delta \( \Delta(0, S_0) \), where \( S_0 \) is the initial underlying asset price.

To find \( V(0, S_0) \) and \( \Delta(0, S_0) \), we can use the formulas for the conditional expectations discussed in the previous sections.
The American option with payoff $\Phi$ and maturity $T$ is usually approximated by a Bermudan option with price $V(0, S_0)$ and delta $\Delta(0, S_0)$, where $S_0$ is the initial underlying asset price.

To find $V(0, S_0)$ and $\Delta(0, S_0)$, we can use the formulas for the conditional expectations discussed in the previous sections.

To this end, we equally subdivide the interval $[0, T]$ into $m(> 1)$ subintervals: $0 = t_0 < t_1 < \cdots < t_m = T$, $t_j = jh$ with step size $h = T/m$. 
Then, $V(0, S_0)$ is approximated by $V_0(S_0)$, where $V_j(S_{jh})$ is defined recursively as follows:
Then, $V(0, S_0)$ is approximated by $V_0(S_0)$, where $V_j(S_{jh})$ is defined recursively as follows:

$$V_m(S_T) = \Phi(S_T),$$
Then, $V(0, S_0)$ is approximated by $V_0(S_0)$, where $V_j(S_{jh})$ is defined recursively as follows:

- $V_m(S_T) = \Phi(S_T)$,
- $V_j(S_{jh}) = \max\{\Phi(S_{jh}), e^{-hr} E \left[ V_{j+1}(S_{(j+1)h}) \mid S_{jh} \right] \}$, for $j = m - 1, \ldots, 1, 0$.
Then, $V(0, S_0)$ is approximated by $V_0(S_0)$, where $V_j(S_{jh})$ is defined recursively as follows:

$$V_m(S_T) = \Phi(S_T),$$

$$V_j(S_{jh}) = \max\{\Phi(S_{jh}), e^{-hr} E\left[V_{j+1}(S_{(j+1)h}) | S_{jh}\right]\},$$

$j = m - 1, \ldots, 1, 0$;

and $\Delta(0, S_0)$ is approximated by

$$\Delta(S_0) = E[\Delta(S_h)],$$
Then, $V(0, S_0)$ is approximated by $V_0(S_0)$, where $V_j(S_{jh})$ is defined recursively as follows:

- $V_m(S_T) = \Phi(S_T)$,
- $V_j(S_{jh}) = \max\{\Phi(S_{jh}), e^{-hr} E[V_{j+1}(S_{(j+1)h})|S_{jh}]\}$, $j = m - 1, \cdots, 1, 0$;
- and $\Delta(0, S_0)$ is approximated by

$$\Delta(S_0) = E[\Delta(S_h)],$$

with

$$\Delta(S_h) = \left\{ \begin{array}{ll}
\frac{\partial \alpha}{\partial \alpha} \Phi(\alpha) |_{\alpha = S_h}, & \text{if } V_1(S_h) < \Phi(S_h) \\
e^{-hr} \frac{\partial \alpha}{\partial \alpha} E[V_2(S_{2h})|S_h = \alpha] |_{\alpha = S_h}, & \text{if } V_1(S_h) > \Phi(S_h) \end{array} \right..$$
Formulas for the conditional expectation \( E \left[ V_{j+1}(S_{(j+1)h}) | S_{jh} \right] \) and the derivative \( \partial_\alpha E \left[ V_2(S_{2h}) | S_h = \alpha \right] \) are given earlier.
Formulas for the conditional expectation $E \left[ V_{j+1}(S_{(j+1)h}) | S_{jh} \right]$ and the derivative $\partial_\alpha E \left[ V_2(S_{2h}) | S_h = \alpha \right]$ are given earlier.

Both $E \left[ V_{j+1}(S_{(j+1)h}) | S_{jh} \right]$ and $\partial_\alpha E \left[ V_2(S_{2h}) | S_h = \alpha \right]$ can be approximated by Monte Carlo or quasi-Monte Carlo simulation methods.
Formulas for the conditional expectation $E \left[ V_{j+1}(S_{(j+1)h}) | S_{jh} \right]$ and the derivative $\partial_\alpha E \left[ V_2(S_{2h}) | S_h = \alpha \right]$ are given earlier.

Both $E \left[ V_{j+1}(S_{(j+1)h}) | S_{jh} \right]$ and $\partial_\alpha E \left[ V_2(S_{2h}) | S_h = \alpha \right]$ can be approximated by Monte Carlo or quasi-Monte Carlo simulation methods.

Thus, we need the samples of the asset prices, which are given by
Formulas for the conditional expectation \( E \left[ V_{j+1}(S_{(j+1)h}) \mid S_{jh} \right] \) and the derivative \( \partial_{\alpha} E \left[ V_2(S_{2h}) \mid S_h = \alpha \right] \) are given earlier.

Both \( E \left[ V_{j+1}(S_{(j+1)h}) \mid S_{jh} \right] \) and \( \partial_{\alpha} E \left[ V_2(S_{2h}) \mid S_h = \alpha \right] \) can be approximated by Monte Carlo or quasi-Monte Carlo simulation methods.

Thus, we need the samples of the asset prices, which are given by

\[
S_{i;t} = S_{i;0} \exp \left( \mu_i Y_t + \sum_{l=1}^{d} c_{il} W_{l;Y_t} \right), \quad i = 1, \ldots, d.
\]

\[
Y_t = \int_{0}^{t} \sum_{l=1}^{d} \sigma_{il} \frac{dW}{\sigma_{il}} + \mu_{t},
\]

where \( \sigma_{il} \) and \( \mu_{t} \) are the volatility and drift of the asset prices, respectively.
The algorithm is given in the following steps:
The algorithm is given in the following steps:

1\(^0\): Generating samples of \( \{Y_t\} \): \( Y_{t_j}^k, j = 1, \ldots, m; \quad k = 1, \ldots, N \).
The algorithm is given in the following steps:

1. Generating samples of \( \{ Y_t \} \): \( Y^k_{t_j}, j = 1, \cdots, m; k = 1, \cdots, N \).

2. Generating samples of \( \{ W_i; Y_t \} \):

\[
W_{i; Y_t}, i = 1, \cdots, d; \quad j = 1, \cdots, m; \quad k = 1, \cdots, N.
\]
The algorithm is given in the following steps:

1°: Generating samples of \( \{ Y_t \} \): \( Y^k_{t_j}, j = 1, \cdots, m; \ k = 1, \cdots, N \).

2°: Generating samples of \( \{ W_i; Y_t \} \):

\[
W_{i; Y_t}, i = 1, \cdots, d; \ j = 1, \cdots, m; \ k = 1, \cdots, N .
\]

3°: Computation of \( \{ S_{i; t} \} \):

\[
S^k_{i; t_j} = S_{i; 0} \exp \left( \mu_i Y^k_{t_j} + \sum_{l=1}^{i} c_{il} W_{l; Y^k_{t_j}} \right) .
\]
4^0: Computation of \( \{ \tilde{S}_{i,t} \} \):

\[
\tilde{S}^k_{i,t_j} = S_{i,0} \exp \left( \mu_i Y^k_{t_j} + p_{i,t_j} + c_{ii} W_{i,Y^k_{t_j}} \right).
\]
4\textsuperscript{0}: Computation of \{\tilde{S}_{i;t}\}:

\[ \tilde{S}_{i;t_j}^k = S_{i;0} \exp \left( \mu_i Y_{t_j}^k + p_{i;t_j} + c_{ii} W_i; Y_{t_j}^k \right). \]

5\textsuperscript{0}: Computation of \{\Delta W_{i,j,k}\}:

\[ \Delta W_{i,j,k} = \Delta W_{t_j,t_{j+1};i}^k = Y_{t_{j+1}}^k W_i; Y_{t_j}^k - Y_{t_j}^k W_i Y_{t_{j+1}}^k + c_{ii} Y_{t_j}^k (Y_{t_{j+1}}^k - Y_{t_j}^k). \]
$6^0$: Computation of $\{V_j(S_{t_j})\}$: use formulas given earlier, where for $j = m - 1, \cdots, 1, 0$, 
6\(^0\): Computation of \( \{ V_j(S_{t_j}) \} \): use formulas given earlier, where for 
\( j = m - 1, \ldots, 1, 0, \)

\[
E \left[ V_{j+1}(S_{t_{j+1}})|S_{t_j} = \alpha \right] |_{\alpha = S_{t_j}^k} = \frac{T_{t_j, t_{j+1}}[V_{j+1}](\alpha)}{T_{t_j, t_{j+1}}[1](\alpha)}
\]
6^0: Computation of \( \{V_j(S_{t_j})\} \): use formulas given earlier, where for \( j = m - 1, \cdots, 1, 0 \),

\[
E \left[ V_{j+1}(S_{t_{j+1}}) | S_{t_j} = \alpha \right] \big|_{\alpha = S_{t_j}^k} = \frac{\prod_{t_j, t_{j+1}} [V_{j+1}](\alpha)}{\prod_{t_j, t_{j+1}} [1](\alpha)}
\]

\[
E \left[ V_{j+1}(S_{t_{j+1}}) \prod_{l=1}^d \frac{H(\tilde{S}_{l; t_j} - \tilde{\alpha}_l)}{\sigma_{ll} Y_{t_j} (Y_{t_{j+1}} - Y_{t_j}) \tilde{S}_{l; t_j}} \Delta W_{Y_{t_j}, Y_{t_{j+1}}; l} \right]
\]

\[
E \left[ \prod_{l=1}^d \frac{H(\tilde{S}_{l; t_j} - \tilde{\alpha}_l)}{\sigma_{ll} Y_{t_j} (Y_{t_{j+1}} - Y_{t_j}) \tilde{S}_{l; t_j}} \Delta W_{Y_{t_j}, Y_{t_{j+1}}; l} \right]
\]
60: Computation of \( \{ V_j(S_{t_j}) \} \): use formulas given earlier, where for 
\( j = m - 1, \ldots, 1, 0, \)

\[
E \left[ V_{j+1}(S_{t_{j+1}}) \middle| S_{t_j} = \alpha \right] \bigg|_{\alpha = S_{t_j}^k} = \frac{\Pi_{t_j, t_{j+1}}[V_{j+1}](\alpha)}{\Pi_{t_j, t_{j+1}}[1](\alpha)}
\]

\[
E \left[ V_{j+1}(S_{t_{j+1}}) \prod_{l=1}^{d} \frac{H(\tilde{S}_l; t_j - \tilde{\alpha}_l)}{\sigma_{l\|} Y_{t_j}(Y_{t_{j+1}} - Y_{t_j}) S_l; t_j} \Delta W_{Y_{t_j}, Y_{t_{j+1}}; l} \right] = \frac{E \left[ \prod_{l=1}^{d} \frac{H(\tilde{S}_l; t_j - \tilde{\alpha}_l)}{\sigma_{l\|} Y_{t_j}(Y_{t_{j+1}} - Y_{t_j}) S_l; t_j} \Delta W_{Y_{t_j}, Y_{t_{j+1}}; l} \right]}{\left( \prod_{l=1}^{d} \frac{H(\tilde{S}_l; t_j - \tilde{\alpha}_l)}{\sigma_{l\|} Y_{t_j}(Y_{t_{j+1}} - Y_{t_j}) S_l; t_j} \right) \Delta W_{Y_{t_j}, Y_{t_{j+1}}; l}}
\]

\[
\sum_{q=1}^{N} V_{j+1}(S_{t_{j+1}}) \prod_{l=1}^{d} \frac{H(\tilde{S}_l^q; t_j - \tilde{S}_l^k; t_j)}{\sigma_{l\|} Y_{t_j}^q(Y_{t_{j+1}}^q - Y_{t_j}^q) S_l^q; t_j} \Delta W_{l, j; q}
\]

\[
\approx \hat{V}_j(S_{t_j}^k) = \frac{\sum_{q=1}^{N} V_{j+1}(S_{t_{j+1}}) \prod_{l=1}^{d} \frac{H(\tilde{S}_l^q; t_j - \tilde{S}_l^k; t_j)}{\sigma_{l\|} Y_{t_j}^q(Y_{t_{j+1}}^q - Y_{t_j}^q) S_l^q; t_j} \Delta W_{l, j; q}}{\sum_{q=1}^{N} \prod_{l=1}^{d} \frac{H(\tilde{S}_l^q; t_j - \tilde{S}_l^k; t_j)}{\sigma_{l\|} Y_{t_j}^q(Y_{t_{j+1}}^q - Y_{t_j}^q) S_l^q; t_j} \Delta W_{l, j; q}}
\]
70: Computation of option price $V(0, S_0)$:

$$V(0, S_0) \approx V_0(S_0) = \max \left( \Phi(S_0), \frac{1}{N} \sum_{k=1}^{N} V_1(S_{t_1}^k) \right),$$

where
70: Computation of option price $V(0, S_0)$:

$$V(0, S_0) \approx V_0(S_0) = \max \left( \Phi(S_0), \frac{1}{N} \sum_{k=1}^{N} V_1(S_{t_1}^k) \right),$$

where

$$V_m(S_{t_m}^k) = \Phi(S_{t_m}^k), \ k = 1, \cdots, N,$$
Algorithms (7)

7^0: Computation of option price \( V(0, S_0) \):

\[
V(0, S_0) \approx V_0(S_0) = \max \left( \Phi(S_0), \frac{1}{N} \sum_{k=1}^{N} V_1(S_{t_1}^k) \right),
\]

where

- \( V_m(S_{t_m}^k) = \Phi(S_{t_m}^k), \ k = 1, \cdots, N, \) and
- \( V_j(S_{t_j}^k) = \max \left( \Phi(S_{t_j}^k), e^{-hr} V_j(S_{t_j}^k) \right), \ j = m - 1, \cdots, 1; \)
8^0: Computation of option delta values
\[ \Delta(S_0) = (\Delta_1(S_0), \ldots, \Delta_d(S_0)) \], where
8^0: Computation of option delta values
\[ \Delta(S_0) = (\Delta_1(S_0), \ldots, \Delta_d(S_0)), \text{ where} \]

\[ \Delta_i(S_0) \approx \overline{\Delta}_i(S_0) = \frac{1}{N} \sum_{k=1}^{N} D_{ik}(S_{t_1}^k), \]
80: Computation of option delta values

\[ \Delta(S_0) = (\Delta_1(S_0), \cdots, \Delta_d(S_0)) \], where

\[ \Delta_i(S_0) \approx \Delta_i(S_0) = \frac{1}{N} \sum_{k=1}^{N} D_{ik}(S_{t_1}^k), \]

\[ D_{ik}(S_{t_1}^k) = \begin{cases} \partial_{\alpha_i} \Phi(\alpha) |_{\alpha=S_{t_1}^k}, & \text{if } U_1(S_{t_1}^k) < \Phi(S_{t_1}^k) \\ e^{-hr} \partial_{\alpha_i} E[V_2(S_{t_2}) | S_{t_1} = \alpha] |_{\alpha=S_{t_1}^k}, & \text{if } U_1(S_{t_1}^k) > \Phi(S_{t_1}^k) \end{cases} \]
and

\[ \partial_{\alpha_i} E \left[ V_2(S_{t_2}) \mid S_{t_1} = \alpha \right] \bigg|_{\alpha = S_{t_1}^k} = \sum_{l=1}^{d} \frac{\hat{S}_{l; t_1}^k}{S_{i; t_1}} \times \]

\[ \mathbb{R}_{t_1, t_2; l} [\Phi](S_{t_1}^k) \mathbb{T}_{t_1, t_2} [1](S_{t_1}^k) - \mathbb{R}_{t_1, t_2; l} [1](S_{t_1}^k) \mathbb{T}_{t_1, t_2} [\Phi](S_{t_1}^k) \]

\[ \left( \mathbb{T}_{t_1, t_2} [1](S_{t_1}^k) \right)^2 \]
\[ \partial_{\alpha_i} E \left[ V_2(S_{t_2}) \mid S_{t_1} = \alpha \right] \big|_{\alpha = S_{t_1}^k} = \sum_{l=1}^{d} \hat{\sigma}_{il} \frac{\tilde{S}_{l;t_1}^k}{S_{i;t_1}^k} \times \frac{\mathbb{R}_{t_1,t_2;l}[\Phi](S_{t_1}^k) \mathbb{T}_{t_1,t_2}[1](S_{t_1}^k) - \mathbb{R}_{t_1,t_2;l}[1](S_{t_1}^k) \mathbb{T}_{t_1,t_2}[\Phi](S_{t_1}^k)}{(\mathbb{T}_{t_1,t_2}[1](S_{t_1}^k))^2} \]

\[ \mathbb{T}_{t_1,t_2}[f](S_{t_1}^k) \approx \frac{1}{N} \sum_{q=1}^{N} f(S_{t_2}^q) \prod_{l=1}^{d} \frac{H(S_{l;t_1}^q - \tilde{S}_{l;t_1}^k)}{\sigma_{ll} Y_{t_1}^q (Y_{t_2}^q - Y_{t_1}^q) \tilde{S}_{l;t_1}^q} \Delta W_{l,1;q}, \]
\[ \mathbb{I}_{t_1, t_2; I}[f](S^k_{t_1}) \approx -\frac{1}{N} \sum_{q=1}^{N} f(S^q_{t_2}) \frac{H(\tilde{S}^q_{l; t_1} - \tilde{S}^k_{l; t_1})}{\sigma_l Y^q_{t_1}(Y^q_{t_2} - Y^q_{t_1})\tilde{S}^q_{l; t_1}} \times \left[ \frac{(\Delta W_{l,1;q})^2}{\sigma_l Y^l_{t_1}(Y^l_{t_2} - Y^l_{t_1})} + \Delta W_{l,1;q} - \frac{Y^2_{t_2}}{\sigma_l} \right] \times \prod_{n=1, n \neq l}^{d} \frac{H(\tilde{S}^q_{n; t_1} - \tilde{S}^k_{n; t_1})}{\sigma_{nn} Y^q_{t_1}(Y^q_{t_2} - Y^q_{t_1})\tilde{S}^q_{n; t_1}} \Delta W_{n,1;q}. \]
When estimating an expectation, $E(X)$, of a r.v. or r. vector, variance or std error or root-mean-square error can be used to measure the ”error”.
When estimating an expectation, $E(X)$, of a r.v. or r. vector, variance or std error or root-mean-square error can be used to measure the "error".

What can be used when estimating the ratio of two expectations $\frac{E(X)}{E(Y)}$?
When estimating an expectation, $E(X)$, of a r.v. or r. vector, variance or std error or root-mean-square error can be used to measure the "error".

What can be used when estimating the ratio of two expectations $\frac{E(X)}{E(Y)}$?

Other type of Levy processes?