American Option Pricing Under Two Stochastic Volatility Processes

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Motivation

- Asset returns are non normal and have fat tails, Blattberg & Gonedes (1974), Platen & Rendek (2008).
- Volatility changes with time and the changes are unpredictable, Scott (1987).
- Though persistent, volatility has a tendency of reverting to a long-run average.
- Adolfsson at. el (2009) extend the Heston model to pricing American Options.
- Multifactor models suggest that asset returns are driven by many unpredictable processes.
- The need to consider two or more stochastic volatility processes.
The Problem Statement

- The risk-neutral dynamics of the driving processes, $S$, $\nu_1$ and $\nu_2$,

\[
\begin{align*}
    dS &= (r - q)Sdt + \sqrt{\nu_1}Sd\tilde{W}_1 + \sqrt{\nu_2}Sd\tilde{W}_2, \\
    dv_1 &= [\kappa_1 \theta_1 - (\kappa_1 + \lambda_1)\nu_1]dt + \rho_{13}\sigma_1 \sqrt{\nu_1}d\tilde{W}_1 \\
        &\quad + \sqrt{1 - \rho_{13}^2}\sigma_1 \sqrt{\nu_1}d\tilde{W}_3, \\
    dv_2 &= [\kappa_2 \theta_2 - (\kappa_2 + \lambda_2)\nu_2]dt + \rho_{24}\sigma_2 \sqrt{\nu_2}d\tilde{W}_2 \\
        &\quad + \sqrt{1 - \rho_{24}^2}\sigma_2 \sqrt{\nu_2}d\tilde{W}_4. \quad (1.1)
\end{align*}
\]

- $\tilde{W}_i$ for $i = 1, \cdots, 4$ are independent Wiener processes.
- $\rho_{13}$ is correlation between $\tilde{W}_1$ and $\tilde{W}_3$ whilst $\rho_{24}$ denotes the correlation between $\tilde{W}_2$ and $\tilde{W}_4$.
- No correlation between $\tilde{W}_1$ & $\tilde{W}_2$, and $\tilde{W}_3$ & $\tilde{W}_4$. 
We set \( \tau = T - t \), \( S = e^x \) and \( C(\tau, x, v_1, \cdots, v_n) \) to be the pricing function. Exploiting the techniques of Jamshidian’s (1992) yields the pricing PDE

\[
\frac{\partial C}{\partial \tau} = \mathcal{L}C - rC + \mathbb{1}_{x \geq \ln b(\tau, v_1, v_2)}(qe^x - rK), \quad (1.2)
\]

where

\[
\mathcal{L} = \left( r - q - \frac{1}{2} v_1 - \frac{1}{2} v_2 \right) \frac{\partial}{\partial x} + \Phi_1 \frac{\partial}{\partial v_1} - \beta_1 v_1 \frac{\partial}{\partial v_1} + \Phi_2 \frac{\partial}{\partial v_2} \\
- \beta_2 v_2 \frac{\partial}{\partial v_2} + \frac{1}{2} v_1 \frac{\partial^2}{\partial x^2} + \frac{1}{2} v_2 \frac{\partial^2}{\partial x^2} + \rho_{13} \frac{\partial^2}{\partial x \partial v_1} \\
+ \rho_{14} \frac{\partial^2}{\partial x \partial v_2} + \frac{1}{2} \sigma_1 v_1 \frac{\partial^2}{\partial v_1^2} + \frac{1}{2} \sigma_2 v_2 \frac{\partial^2}{\partial v_2^2}, \quad (1.3)
\]

and

\[
\Phi_1 = \kappa_1 \theta_1, \quad \Phi_2 = \kappa_2 \theta_2, \quad \beta_1 = \kappa_1 + \lambda_1 \quad \text{and} \quad \beta_2 = \kappa_2 + \lambda_2.
\]
Eqn (1.3) is solved subject to the initial and boundary conditions,

\[
C(0, x, v_1, v_2) = (e^x - K)^+, \quad -\infty < x < \infty, \quad (1.4)
\]
\[
b(\tau, v_1, v_2) - K = C(\tau, b(\tau, v_1, v_2), v_1, v_2). \quad (1.5)
\]

Smooth pasting condition can also be imposed depending on the particular problem considered.

After effecting the transformation \( S_j = e^{x_j} \) the transition density function \( U(\tau, x, v_1, v_2) \) for the SDE system (1.1) is a solution of the backward Kolmogorov PDE,

\[
\frac{\partial U}{\partial \tau} = \mathcal{L} U \quad (1.6)
\]

Equation (1.6) is to be solved subject to the initial condition,

\[
U(0, x, v_1, v_2; x_0, v_{1,0}, v_{2,0}) = \delta(x - x_0)\delta(v_1 - v_{1,0})\delta(v_2 - v_{2,0}). \quad (1.7)
\]
Duhamel’s principle states that the solution to the one dimensional inhomogeneous parabolic PDE of the form,

\[
\frac{\partial U}{\partial \tau} = \mathcal{L} U + f(\tau, x),
\]

subject to the initial condition,

\[
U(0, x) = \phi(x),
\]

can be represented as,

\[
U(\tau, x) = \int_{-\infty}^{\infty} \phi(y)U(\tau, x - y)dy + \int_{0}^{\tau} \int_{-\infty}^{\infty} f(\xi, y)U(\tau - \xi, x - y)dyd\xi.
\]

Here, \(\mathcal{L}\) is a parabolic partial differential operator.
By use of Duhamel’s principle, the solution of the American call option pricing PDE (1.3) can be represented as,

\[
C(\tau, x, v_1, v_2) = C_E(\tau, x, v_1, v_2) + C_P(\tau, x, v_1, v_2), \tag{1.9}
\]

where,

\[
C_E(\tau, x, v_1, v_2) = e^{-r\tau} \int_0^\infty \int_0^\infty \int_{-\infty}^{\infty} (e^u - K)^+ \\
\times U(\tau; x, v_1, v_2; u, w_1, w_2) \, du \, dw_1 \, dw_2,
\]

\[
C_P(\tau, x, v_1, v_2) = \int_0^\tau e^{-r(\tau - \xi)} \int_0^\infty \int_0^\infty \int_{\ln b(\xi, w_1, w_2)}^{\infty} (qe^u - rK) \\
\times U(\tau - \xi; x, v_1, v_2; u, w_1, w_2) \, du \, dw_1 \, dw_2 \, d\xi.
\]

The 1\textsuperscript{st} part of (1.9) is the European Option component and the 2\textsuperscript{nd} is the Early Exercise premium.
A Fourier transform is applied to the log $S$ variable followed by Laplace transforms to the $v$ variables of the PDE (1.6) and solving the resulting system of PDE by the method of characteristics.

Once the system of PDEs is solved, we use the tabulated results in Abramowitz and Stegun (1964) to find the inverse Laplace transform of the resulting solution.

Application of the inverse Fourier transform to the resulting solution yields the transition density function as,

$$U(\tau, \log S, v_1, v_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta \ln S} \left\{ e^{i\eta S_0 - i\eta(r-q)} \prod_{j=1}^{2} \exp \left\{ \left( \frac{\Theta_j - \Omega_j}{\sigma_j^2} \right) (v_j - v_{j,0} + \Phi_j \tau) \right\} \right.$$ 

$$\times \exp \left\{ - \left( \frac{2\Omega_j}{\sigma_j^2 (e^\Omega_j \tau - 1)} \right) (v_{j,0} e^{\Omega_j \tau} + v_j) \right\} \frac{2\Omega_j e^{\Omega_j \tau}}{\sigma_j^2 (e^\Omega_j \tau - 1)} \left( \frac{v_{j,0} e^{\Omega_j \tau}}{v_j} \right)^{\frac{\Phi_j}{\sigma_j^2} - \frac{1}{2}}$$ 

$$\times I_{\frac{2\Phi_j}{\sigma_j^2} - 1} \left( \frac{4\Omega_j}{\sigma_j^2 (e^\Omega_j \tau - 1)} (v_j v_{j,0} e^{\Omega_j \tau})^{\frac{1}{2}} \right) \} \right\} d\eta.$$  

(1.10)
The American Call Option Price

By letting $V(\tau, S, v_1, v_2) \equiv C(\tau, \log S, v_1, v_2)$ and approximating the early exercise boundary with the expression,

$$\ln b(\tau, v_1, v_2) \approx b_0(\tau) + b_1(\tau)v_1 + b_2(\tau)v_2,$$  \hspace{1cm} (1.11)

the value of the American call option can be expressed as,

$$V(\tau, S, v_1, v_2) \approx V_E(\tau, S, v_1, v_2) + V_A(\tau, S, v_1, v_2),$$  \hspace{1cm} (1.12)

The component on the RHS can be represented as,

$$V_E(\tau, S, v_1, v_2) = e^{-q\tau SP_1(\tau, S, v_1, v_2; K)} - e^{-r\tau KP_2(\tau, S, v_1, v_2; K)},$$  \hspace{1cm} (1.13)

where,

$$P_j(\tau, S, v_1, v_2; K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{g_j(\tau, S, v_1, v_2; \eta)e^{-i\eta \ln K}}{i\eta} \right) d\eta,$$  \hspace{1cm} (1.14)

for $j = 1, 2.$
The early exercise premium is given as

\[
V^A_P(\tau, S, v_1, v_2) = \int_0^{\tau} \left[ qe^{-q(\tau-\xi)} S\bar{P}^A(\tau - \xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi))
\right.
\]

\[
- re^{-r(\tau-\xi)} K\bar{P}^A_2(\tau - \xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi))]d\xi, \quad (1.15)
\]

with,

\[
\bar{P}^A_j(\tau - \xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi)) = \frac{1}{2}
\]

\[
+ \frac{1}{\pi} \int_0^{\infty} \text{Re} \left( \frac{\bar{g}_j(\tau - \xi, S, v_1, v_2; \eta, b_1(\xi), b_2(\xi))e^{-i\eta b_0(\xi)}}{i\eta} \right) d\eta,
\]

for \( j = 1, 2. \)
Iterative Equations

Given equation (1.11), the value – matching condition can be expressed as,

$$e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2} - K = V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2).$$

(1.17)

The implicit time functions are found by solving the system,

$$b_0(\tau) = \ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_1(\tau)v_1 - b_2(\tau)v_2,$$

$$b_1(\tau) = \frac{1}{v_1} \left( \ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_0(\tau) - b_2(\tau)v_2 \right),$$

$$b_2(\tau) = \frac{1}{v_2} \left( \ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_0(\tau) - b_1(\tau)v_1 \right).$$

(1.18)
Numerical Implementation

- In implementing equations (1.12) and the system (1.18), we treat the American option as a Bermudan option.
- The time interval is partitioned into $M$ – equally spaced subintervals of length $h = T/M$.
- It has been shown in Kim (1990) that the early exercise boundary at maturity is,

$$b(0, v_1, v_2) = \max \left( \frac{r}{q} K, K \right).$$  \hfill (1.19)

By comparing coefficients, we can readily deduce that,

$$b_0(0) = \max \left( K, \frac{r}{q} K \right), \quad b_1(0) = 0, \quad \text{and,} \quad b_2(0) = 0.$$  \hfill (1.20)
Numerical Implementation cont...

- The discretized version of the pricing function is,
  \[ V(hm, S, v_1, v_2) = V_E(hm, S, v_1, v_2) + V_P(hm, S, v_1, v_2). \]  
  \[ (1.21) \]

- At each time we determine the three unknown boundary terms, \( b_0^m = b_0(hm), b_1^m = b_1(hm) \) and \( b_2^m = b_2(hm) \).

- We solve iteratively,
  \[ b_{0,k}^m = \ln[V(hm, e^{b_{0,k}^m+b_{1,k-1}^m v_1+b_{2,k-1}^m v_2}, v_1, v_2) + K] - b_{1,k-1}^m v_1 - b_{2,k-1}^m v_2, \]
  \[ b_{1,k}^m = \frac{1}{v_1} \left( \ln[V(hm, e^{b_{0,k}^m+b_{1,k}^m v_1+b_{2,k-1}^m v_2}, v_1, v_2) + K] - b_{0,k}^m - b_{2,k-1}^m v_2 \right), \]
  \[ b_{2,k}^m = \frac{1}{v_2} \left( \ln[V(hm, e^{b_{0,k}^m+b_{1,k}^m v_1+b_{2,k}^m v_2}, v_1, v_2) + K] - b_{0,k}^m - b_{1,k}^m v_1 \right). \]  
  \[ (1.22) \]

- We continuously repeat the iterative process until a tolerance level is reached.
# Numerical Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>$v_1$ – Parameter</th>
<th>Value</th>
<th>$v_2$ – Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>100</td>
<td>$\theta_1$</td>
<td>6%</td>
<td>$\theta_2$</td>
<td>8%</td>
</tr>
<tr>
<td>$r$</td>
<td>3%</td>
<td>$\kappa_1$</td>
<td>3</td>
<td>$\kappa_2$</td>
<td>4</td>
</tr>
<tr>
<td>$q$</td>
<td>5%</td>
<td>$\sigma_1$</td>
<td>10%</td>
<td>$\sigma_2$</td>
<td>11%</td>
</tr>
<tr>
<td>$T$</td>
<td>0.5</td>
<td>$\rho_{12}$</td>
<td>±0.5</td>
<td>$\rho_{13}$</td>
<td>±0.5</td>
</tr>
<tr>
<td>$M$</td>
<td>200</td>
<td>$\lambda_1$</td>
<td>0</td>
<td>$\lambda_2$</td>
<td>0</td>
</tr>
<tr>
<td>$v_{1\text{max}}$</td>
<td>20%</td>
<td></td>
<td></td>
<td>$v_{2\text{max}}$</td>
<td>20%</td>
</tr>
</tbody>
</table>

**Table:** Parameters used for the American call option. The $v_1$ column contains are parameters for the first variance process whilst the $v_2$ column contains parameters for the second variance process.
Free Surface of the American Call Option

Figure: Early Exercise Surface of the American Call option when $v_2 = 0.67\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. All other parameters are as presented in Table 1.1.
Effects of Stochastic Volatility on the Early Exercise Boundary

Figure: Exploring the effects of stochastic volatility on the early exercise boundary of the American call option for varying correlation coefficients when $\sigma_{GBM} = 0.3742$, $\nu_1 = 6\%$ and $\nu_2 = 8\%$. All other parameters are provided in Table 1.1.
Figure: Comparing early exercise boundaries from the MOL and Numerical integration approach when the two instantaneous variances are fixed. Here, \( \nu_1 = 0.67\% \), \( \nu_2 = 13.33\% \), \( \rho_{13} = 0.5 \) and \( \rho_{24} = 0.5 \) with all other parameters as given in Table 1.1.
**Figure:** Option prices from the Geometric Brownian motion minus option prices from the Stochastic volatility model for varying correlation coefficients. Here, $\sigma_{GBM} = 0.3742$, $\nu_1 = 6\%$ and $\nu_2 = 8\%$ with all other parameters provided in Table 1.1.
**Price Comparisons**

<table>
<thead>
<tr>
<th>$S$</th>
<th>Numerical Integration</th>
<th>MOL</th>
<th>GBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.2036</td>
<td>0.2029</td>
<td>0.1850</td>
</tr>
<tr>
<td>80</td>
<td>2.4088</td>
<td>2.400</td>
<td>2.4154</td>
</tr>
<tr>
<td>100</td>
<td>9.8082</td>
<td>9.7918</td>
<td>9.9452</td>
</tr>
<tr>
<td>120</td>
<td>23.1069</td>
<td>23.0920</td>
<td>23.3006</td>
</tr>
<tr>
<td>140</td>
<td>40.4756</td>
<td>40.4686</td>
<td>40.5922</td>
</tr>
<tr>
<td>160</td>
<td>60</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>180</td>
<td>80</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>200</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Table: American call option price comparisons when $\nu_1 = 0.67\%$, $\nu_2 = 13.33\%$, $\rho_{13} = 0.5$, $\rho_{24} = 0.5$. We have taken GBM volatility to be $\sigma_{GBM} = 0.3741657$ and this is found by using the formula $\sigma_{GBM} = \sqrt{\theta_1 + \theta_2}$. 
Summary

- We have derived the integral representation of an American call option when the underlying asset is driven by two stochastic variance processes.
- An explicit form of the transition density function has been provided.
- We approximated the three dimensional early – exercise boundary by a multivariate log – linear function.
- Numerical results and comparisons with alternative methods have been presented.
Possible Extensions

- Incorporating more than two stochastic volatility processes.
- Generalizing to Multiple assets under Multiple stochastic volatility.
- Performing empirical studies on multifactor models.