Pricing Options on Realized Variance in Lévy Models

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based on joint work with Johannes Muhle-Karbe

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Let $S$ denote a discounted asset, and $X$ its logarithm.

The **annualized realized variance** of $X$ over the period $[0, T]$ subdivided into $n$ business days $0 = t_0 < \cdots < t_n = T$ is given by

$$RV_n(T) = \frac{1}{T} \sum_{k=1}^{n} \left( \log \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 = \frac{1}{T} \sum_{k=1}^{n} \left( X_{t_k} - X_{t_{k-1}} \right)^2$$

A considerable number of financial instruments use realized variance as an underlying: **variance swap**, **volatility swap**, **calls/puts on realized variance**
Standard pricing approach: substitute annualized quadratic variation $QV(T) = \frac{1}{T}[X, X]_T$ for realized variance.

$$RV_n(T) \approx QV(T)$$

Quadratic variation is the limit in probability of realized variance, when $T$ stays fixed and the number of increments $n$ tends to infinity.
The approximation via quadratic variation works well for claims with (approximately) linear payoffs: variance swaps, volatility swaps.

See Bühler [2006], Sepp [2008], Broadie and Jain [2008]

The approximation is not sufficient for claims with non-linear payoffs like calls/puts and for maturities shorter than 3 months.

See Bühler [2006], Gatheral [2008].
ATM call in the Heston model. Plot taken from Bühler [2006].
This talk addresses the following questions:

1. How big is the *discretization gap* between options on quadratic variation (QV) and realized variance (RV)?
2. How can options on the realized variance be valuated exactly?

We focus on ATM calls, i.e., options with payoff

\[ (RV_n(T) - \mathbb{E}[RV_n(T)])^+ \]

where \( \mathbb{E}[RV_n(T)] \) is the swap rate.
The Discretization Gap

As a proxy for the short-time behavior of options on realized variance we use

\[
\lim_{T \to 0} \mathbb{E} \left[ (RV_1(T) - \mathbb{E}[RV_1(T)])^+ \right];
\]

for options on quadratic variation we use

\[
\lim_{T \to 0} \mathbb{E} \left[ (QV(T) - \mathbb{E}[QV(T)])^+ \right].
\]

The discretization gap is the difference between the two:

\[
\lim_{T \to 0} \left\{ \mathbb{E} \left[ (RV_1(T) - \mathbb{E}[RV_1(T)])^+ \right] - \mathbb{E} \left[ (QV(T) - \mathbb{E}[QV(T)])^+ \right] \right\}.
\]

Note that \( RV_1(T) \) is the realized variance over a single business day, i.e. \( RV_1(T) = \frac{1}{T} X_T^2 \)
We assume that the underlying $X$ follows a Lévy process:

$X$ can be characterized by its Lévy triplet $(b, \sigma^2, F)$, or by its Lévy exponent

$$\psi(u) = bu + \frac{\sigma^2}{2} u^2 + \int (e^{ux} - 1 - uh(x)) F(dx).$$

We also assume that the first two moments of $X$ exist. In this case $X$ has a decomposition

$$X_t = bt + \sigma W_t + L_t$$

where $L$ is a centered pure-jump process of finite variance, and $W$ an independent Brownian motion.
Results for Lévy processes – Quadratic variation

Theorem (K.-R. and Muhle-Karbe (2010))

For a Lévy process $X$ a call on quadratic variation satisfies

$$\lim_{T \to 0} \mathbb{E} \left[ (QV(T) - \mathbb{E}[QV(T)])^+ \right] = \nu^2,$$

where $\nu^2 = \int x^2 F(dx)$.

Note: $\nu^2$ is the variance of the pure jump component $L$. 
Theorem (K.-R. and Muhle-Karbe (2010))

For a Lévy process $X$ a call on realized variance satisfies

$$\lim_{T \to 0} \mathbb{E} \left[ (RV_1(T) - \mathbb{E}[RV_1(T)])^+ \right] = \sigma^2 P \left( \frac{\nu^2}{\sigma^2} \right) + \nu^2 Q \left( \frac{\nu^2}{\sigma^2} \right),$$

where $\nu^2 = \int x^2 F(dx)$ and $P(r)$ resp. $Q(r)$ are strictly decreasing resp. increasing functions on $[0, \infty)$, given by

$$P(r) = \sqrt{\frac{2(1 + r)}{\pi \exp(1 + r)}}, \quad \text{and} \quad Q(r) = 2\Phi(\sqrt{1 + r}) - 1,$$

with $\Phi(.)$ denoting the standard normal distribution function.
Pure diffusion – no jumps:

\[
\lim_{T \to 0} \mathbb{E} \left[ (QV(T) - \mathbb{E}[QV(T)])^+ \right] = 0
\]

\[
\lim_{T \to 0} \mathbb{E} \left[ (RV_1(T) - \mathbb{E}[RV_1(T)])^+ \right] = \sqrt{\frac{2}{\pi e}} \sigma^2 \approx 0.48 \sigma^2
\]

Under mild conditions these results also hold in pure-diffusion models with stochastic volatility (but without leverage effect).
Special cases – Pure jump

Pure jump process – no diffusion:

\[
\lim_{T \to 0} \mathbb{E} \left[ (QV(T) - \mathbb{E}[QV(T)])^+ \right] = \nu^2
\]

\[
\lim_{T \to 0} \mathbb{E} \left[ (RV_1(T) - \mathbb{E}[RV_1(T)])^+ \right] = \nu^2
\]

The discretization gap vanishes completely in pure-jump models!

In **true jump-diffusion** models the interaction between jump and diffusion component is surprisingly complex.
Numerical results for 2 Lévy-based models with 3 different parameter sets:

- The Kou model is a jump-diffusion model with double-exponentially distributed jump sizes.
- The CGMY model is a pure jump model introduced by Carr, Geman, Madan and Yor.
- We use calibrated parameter sets from Sepp [2008] and Carr et al. [2005] respectively. For the Kou model we also look at the effect of reducing the diffusion volatility $\sigma$ from 0.3 to 0.2.
Figure: ATM call prices on normalized quadratic variation resp. realized variance in the Kou model for $\sigma = 0.3$. The analytic short-time limits from the corresponding theorems are 0.0718 resp. 0.0980.
Figure: ATM call prices on normalized quadratic variation resp. realized variance in the Kou model for $\sigma = 0.2$. The analytic short-time limits from the corresponding theorems are 0.0706 resp. 0.0773.
Figure: ATM call prices on normalized quadratic variation resp. realized variance in the CGMY model. The discretization gap vanishes as predicted.
How did we produce the numerical results?

- Monte-Carlo simulation can be problematic for Lévy processes: transition density not known in closed form, jumps may have infinite arrival rate, etc.

- For quadratic variation Fourier-based methods have been described in the previous talk.

- For realized variance we propose analogous methods in [K.-R. and Muhle-Karbe (2010)].
Fourier Pricing

Suppose the Laplace transform \( \mathbb{E} \left[ e^{-uX_t^2} \right] \) of the squared Lévy process is known in the half plane \( \mathcal{H}_+ = \{ u \in \mathbb{C} : \text{Re}(u) \geq 0 \} \).

Applying the Fourier-pricing approach of Carr & Madan yields:

Fourier Pricing for calls on realized variance

\[
\mathbb{E} \left[ (RV_n(T) - K)^+ \right] = \\
= \mathbb{E} [RV_n(T)] - K + \frac{1}{\pi} \int_{\alpha}^{\alpha+i\infty} \text{Re} \left( \frac{e^{Ku}}{u^2} \mathbb{E} \left[ \exp \left( -uX_\delta^2 \right) \right]^n \right) du
\]

where \( \alpha > 0 \) and \( \delta = T/n \).
Theorem (K.-R. and Muhle-Karbe (2010))

Let $X_t$ be a Lévy process, whose characteristic exponent $\psi(u)$ satisfies a mild analyticity condition. Let $Z$ be an independent standard normal random variable. Then

$$\mathbb{E} \left[ e^{-uX_t^2} \right] = \mathbb{E} \left[ e^{t\psi(iZ\sqrt{2u})} \right]$$

holds for all $u$ in the complex half-plane $\mathcal{H}_+ = \{ u : \text{Re}(u) > 0 \}$.

- Replaces the integration with respect to the law of the Lévy process by an integration with respect to a normal distribution.
- The analyticity condition holds e.g. for the Kou and the Merton model, the NIG, the Variance Gamma and the CGMY process.
In many cases quadratic variation is not a good proxy for realized variance, when pricing of call/put options on realized variance is concerned.

The difference in prices is most pronounced in diffusion models, decreases when jumps are added, and vanishes completely in pure-jump models.

We have presented methods for exact valuation of options on realized variance by Fourier methods in the context of exponential-Lévy models.

Extensions to stochastic volatility models with jumps are work in progress.
Thank you for your attention!

For details see:


Generalization of the realized variance result

Theorem (Generalized short-time limit)

For a Lévy process $X$ a call on realized variance satisfies

$$\lim_{T \to 0} \mathbb{E} \left[ (RV_n(T) - k \mathbb{E}[RV_n(T)])^+ \right] =$$

$$\sigma^2 P_{k,n} \left( \frac{v^2}{\sigma^2} \right) + \left( \sigma^2 (k - 1) + v^2 k \right) Q_{k,n} \left( \frac{v^2}{\sigma^2} \right),$$

where $v^2 = \int x^2 F(dx)$ and $P_{k,n}(r)$ and $Q_{k,n}(r)$ are given by

$$P_{k,n}(r) = \frac{2/n}{\Gamma(n/2)} \left( \frac{nk(1+r)}{2 \exp(k(1+r))} \right)^{n/2}$$

$$Q_{k,n}(r) = \gamma_0(n/2, nk(1+r)/2)$$

with $\gamma_0(., .)$ denoting the regularized incomplete Gamma function.