A factor contagion model for portfolio credit derivatives with interacting recovery rate

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Introduction

Goal
- Valuation of portfolio credit derivatives when there is a contagion effect

Outline
- Marshall-Olkin copula & Contagion model
- Distribution of the $k$th default time
- Portfolio loss distribution
- Interacting recovery rate
Introduction

Copula Model

- Li(2000) : Gaussian copula function approaches
- Laurent & Gregory(2005), Andersen & Sidenius(2003), Bastide et al.(2007) : pricing method with various copula functions
- Andersen et al.(2003), Hull & White(2004) : recursive techniques to derive loss distributions

Contagion Model

- Davis & Lo(2001) : infection model
- Jarrow & Yu(2001) : primary-secondary framework

Default Probability & Recovery Rate

- Altman et al.(2005) : relations between default probabilities and recovery rates
**Bivariate Marshall-Olkin copula**

**Assumptions**

- \( Z_1 \sim \exp(\lambda_1), Z_2 \sim \exp(\lambda_2), Z \sim \exp(\lambda) \): independent
- \( X_1 = \min(Z, Z_1) \) and \( X_2 = \min(Z, Z_2) \)
- \( s_i(x_i) = e^{-\left(\lambda_i + \lambda\right)x_i} \), \( s(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda \max(x_1, x_2)} \)
  - marginal and joint survival functions of \( X_1 \) and \( X_2 \)

**Bivariate Marshall-Olkin copula**

- By Sklar’s theorem, there exists a unique survival copula \( C \) satisfying
  \[
s(x_1, x_2) = C(s_1(x_1), s_2(x_2)).
\]
- Marshall-Olkin copula is given by
  \[
  C(u_1, u_2) = \min(u_2 u_1^{1-\theta_1}, u_1 u_2^{1-\theta_2}),
  \]
  where \( \theta_i = \frac{\lambda_i}{\lambda + \lambda_i} \).
One factor Marshall-Olkin copula model

Assumptions

- $V_0 \sim \exp(\alpha) :$ systematic factor, $0 \leq \alpha \leq 1$
  
  $V_i \sim \exp(1 - \alpha) :$ idiosyncratic factor, $i = 1, \ldots, n$
  
- $V_0$ and $V_i$ are independent
  
- $\overline{V}_i = \min(V_0, V_i) \sim \exp(1)$

Definition of default times

- The default time $\tau_i$ of name $i$ is defined as
  
  $$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(s) ds \geq \overline{V}_i \right\}. $$

- The distribution function of the $k$th default time is
  
  $$F^{(k)}(t) = \int_0^\infty \mathbb{P}(\tau^k \leq t | V_0 = v) \phi(v) dv, $$

  where $\phi$ is the probability density function of $V_0$ and $\tau^k$ is the $k$th default time.
Default intensity and default times

\[ \lambda_i(t) = a(t) + c(t) \sum_{j=1, j \neq i}^{n} 1_{\{\tau_j \leq t\}} : \text{default intensity} \]

Assuming that the \( j \)th default occurs, we define the following random variables conditional on \( V_0 = v \),

\[ \tau_i(v) = \inf \left\{ t > 0 : \int_0^t (a(s) + jc(s)) ds \geq \min(v, V_i) \right\} \]

: conditional default time after \( j \)th default

\[ \tau_{i:n-j}(v) : \text{ith order statistic of } \tau_1(v), \ldots, \tau_{n-j}(v), \text{i.e.,} \]

\[ \tau_{1:n-j}(v) < \cdots < \tau_{i:n-j}(v) < \cdots < \tau_{n-j:n-j}(v). \]
The $k$th default time $\tau^k(v)$ conditional on $V_0 = v$ is defined by

$$\tau^k(v) = \sum_{j=1}^{k} \tau_{1:n-j+1}(v).$$

The distribution function of the $k$th default time can be written as

$$F^{(k)}(t) = \int_0^\infty \mathbb{P}\left( \sum_{j=1}^{k} \tau_{1:n-j+1}(v) \leq t \right) \phi(v) \, dv,$$

where $\phi$ is the probability density function of $V_0$. 
Theorem (Distribution function of $k$th default time - General case)

Let the default time $\tau_i$ and its default intensity $\lambda_i(t)$ be defined as above. Let

$$Q_i(t) = \int_0^t (a(x) + ic(x))dx$$

and

$$f_{\ell,n}(t) = (1 - \alpha)(n - \ell + 1)Q'_{\ell-1}(t)e^{-(1-\alpha)(n-\ell+1)Q_{\ell-1}(t)}.$$ 

Let $\zeta_i$ be the function whose Laplace transform is given by

$$\hat{\zeta}_i(s) = \frac{1}{s^{i+1}} \prod_{\ell=1}^{i} \mathbb{E} f_{\ell,n} \left( \frac{X}{s} \right),$$

where $X$ be a unit exponential random variable.

Then the distribution function of $\tau^k$ is

$$F^{(k)}(t) = \zeta_k(t)e^{-\alpha Q_{k-1}(t)} + \sum_{i=1}^{k-1} \zeta_i(t)(e^{-\alpha Q_{i-1}(t)} - e^{-\alpha Q_i(t)}) + 1 - e^{-\alpha Q_0(t)}.$$
Theorem (Distribution function of $k$th default time - Special case)

Let $a(t) \equiv a > 0$ and $c(t) \equiv c \geq 0$ be constants. Let $Q_i(t) = (a + ic)t$ and let $\zeta_i$ be the function which is given by

$$\zeta_i(t) = 1 - \sum_{\ell=1}^{i} A_{\ell,i} \exp \left( - \frac{1 - \alpha}{p_{\ell,n}} t \right)$$

where

$$A_{\ell,i} = \frac{p_{j,n}^{i-1}}{\prod_{j=1, j\neq \ell}^{i} (p_{\ell,n} - p_{j,n})} \quad \text{and} \quad p_{\ell,n} = \frac{1}{(n - \ell + 1)q_{\ell-1}}.$$

Then the distribution function of $\tau_k$ is

$$F^{(k)}(t) = \zeta_k(t)e^{-\alpha Q_{k-1}(t)} + \sum_{i=1}^{k-1} \zeta_i(t)(e^{-\alpha Q_{i-1}(t)} - e^{-\alpha Q_i(t)}) + 1 - e^{-\alpha Q_0(t)}.$$
Corollary (The number of defaults up to time $t$)

Let $N(t)$ be the number of defaults up to time $t$. Then

$$
\mathbb{P}(N(t) = k) = \begin{cases} 
(1 - \zeta_1(t))e^{-\alpha Q_0(t)} & \text{if } k = 0 \\
(\zeta_k(t) - \zeta_{k+1}(t))e^{-\alpha Q_k(t)} & \text{if } k = 1, \ldots, n - 1 \\
F^{(n)}(t) & \text{if } k = n
\end{cases}
$$

where $\zeta_k(t)$, $Q_k(t)$ and $F^{(n)}(t)$ are given in the previous theorems.
Now we compute premiums of portfolio credit derivatives.

We use the following notations:

- \( R \) : recovery rate
- \( T \) : maturity of the contracts
- \( B(0, t) \) : price of risk-free zero coupon bond with maturity \( t \)
- \( t_i \) : premium payment dates, \( 0 = t_0 < \cdots < t_N = T \) and \( \Delta_{i-1,i} = t_i - t_{i-1} \)

Proposition (Premium of \( k \)-th-to-default swap)

The annualized premium \( s_k \) of a \( k \)-th-to-default swap is equal to

\[
\begin{align*}
    s_k &= \frac{(1 - R) \int_0^T B(0, t) dF^{(k)}(t)}{\sum_{i=1}^N \left\{ \Delta_{i-1,i} B(0, t_i) (1 - F^{(k)}(t_i)) + \Delta_{i-1,i} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) B(0, t) dF^{(k)}(t) \right\}}.
\end{align*}
\]
Define a cumulative percentage loss $L(t)$ on the homogeneous reference portfolio up to time $t$

$$L(t) = \frac{1 - R}{n} \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}} = \frac{1 - R}{n} N(t).$$

Consider a CDO tranche with an attachment point $A$ and a detachment point $B$ where $0 \leq A < B \leq 1$. The percentage loss $L(t, A, B)$ on the tranche $[A, B]$ up to time $t$ is defined by

$$L(t, A, B) = \frac{\max(L(t) - A, 0) - \max(L(t) - B, 0)}{B - A}.$$
Proposition (Premium of single tranche CDO)

The premium \( s_{A,B} \) of the tranche with an attachment point \( A \) and a detachment point \( B \) is equal to

\[
 s_{A,B} = \frac{B(0, T) \mathbb{E}[L(T, A, B)] + \int_0^T f(0, s)B(0, s)\mathbb{E}[L(s, A, B)]ds}{\sum_{i=1}^{N_i} \Delta_{i-1,i}B(0, t_i)(1 - \mathbb{E}[L(t_i, A, B)])}
\]

where

\[
 \mathbb{E}[L(t, A, B)] = 1 - \frac{1}{B - A} \int_A^B \left\lfloor \frac{n - R}{x} \right\rfloor \sum_{\ell=0}^{\lfloor n - 1 \rfloor} \mathbb{P}(N(t) = \ell)dx,
\]

\( \lfloor y \rfloor \) is the largest integer not greater than \( y \) and \( f(0, t) \) is the spot forward rate.
Define an interacting recovery rate $R(t)$ by

$$R(t) = R_0 - \gamma \sum_{k=1}^{n} 1_{\{\tau^k \leq t\}}.$$

In this case, the cumulative percentage loss is given by

$$\tilde{L}(t) = \frac{1}{n} \sum_{k=1}^{n} (1 - (R_0 - k\gamma)) 1_{\{\tau^k \leq t\}}.$$

and the loss on a tranche $[A, B]$ is

$$\tilde{L}(t, A, B) = \max(\tilde{L}(t) - A, 0) - \max(\tilde{L}(t) - B, 0) \cdot \frac{B - A}{B - A}.$$
Proposition (Interacting recovery rate)

The premium $\tilde{s}_{A,B}$ of the tranche with an attachment point $A$ and a detachment point $B$ is equal to

$$\tilde{s}_{A,B} = \frac{B(0, T)E[\tilde{L}(T, A, B)] + \int_{0}^{T} f(0, s)B(0, s)E[\tilde{L}(s, A, B)]ds}{\sum_{i=1}^{N} \Delta_{i-1,i} B(0, t_i)(1 - E[\tilde{L}(t_i, A, B)])}$$

where

$$E[\tilde{L}(t, A, B)] = 1 - \frac{1}{B - A} \int_{A}^{B} \sum_{\ell=0}^{|\beta(x)|} P(N(t) = \ell) dx$$

and

$$\beta(x) = \frac{1}{2\gamma} \left( - (2(1 - R_0) + \gamma) + \sqrt{(2(1 - R_0) + \gamma)^2 + 8\gamma nx} \right).$$
Numerical results

Default intensity

Assume that the default intensity is given by

$$\lambda_i(t) = ae^{-\delta t} \left( 1 + \theta \sum_{j=1, j\neq i}^{n} 1\{\tau_j \leq t\} \right).$$

- $a$ : the base default intensity
- $\delta$ : the rate of exponential decay
- $\theta$ : the level of contagion.
Distribution of default times

Distributions of $k$th default time

Figure: Probability densities of the 5th default times with $\alpha = 0$ (left) and $\alpha = 0.3$ (right)
Figure: Probabilities of the number of defaults up to time $t = 5$ with $\alpha = 0, 0.3, 0.6, 0.9$ for the contagion level $\theta = 0$ (left) and $\theta = 0.1$ (right).
Figure: Premiums of $k$th-to-default swaps against contagion levels $0 \leq \theta \leq 3$ with $\alpha = 0$ (left) and $\alpha = 0.5$ (right) for $k = 2, 4, 6, 8, 10$
**Figure**: Premiums of $k$th-to-default swaps against correlation $0 \leq \alpha \leq 1$ with $T = 5$ for $k = 1$ (upper left), $k = 2$ (upper right), $k = 3$ (lower left) and $k = 4$ (lower right)
Figure: Premiums of STCDOs against contagion levels $0 \leq \theta \leq 3$ with $\alpha = 0$ (left) and $\alpha = 0.5$ (right) for $[A, B]=[0\%, 3\%], [3\%, 6\%], [6\%, 9\%], [9\%, 12\%]$
STCDO-Correlation

Figure: Premiums of STCDOs against correlations $0 \leq \alpha \leq 1$ with $[A, B]=[0\%, 3\%]$ (upper left), $[3\%, 6\%]$ (upper right), $[6\%, 9\%]$ (lower left), $[9\%, 12\%]$ (lower right)
STCDO-Interacting recovery rate

**Figure:** Premiums of STCDOs against $0 \leq \gamma \leq 0.005$ with $\theta = 0$, $\alpha = 0$ (upper left), $\theta = 0$, $\alpha = 0.5$ (upper right), $\theta = 0.5$, $\alpha = 0$ (lower left) and $\theta = 0.5$, $\alpha = 0.5$ (lower right)
A homogeneous reference portfolio with correlation risks as well as contagion effect

One factor contagion model with Marshall-Olkin copula

A simple and efficient method for finding the distribution of the $k$th default time and pricing portfolio credit derivatives

The reference portfolio with interacting recovery rates

The relationship between premiums and parameters such as default correlation and the level of contagion
Thank you!