Stochastic Volatility and Jumps: Exponentially Affine Yes or No?
An Empirical Analysis of S&P500 Dynamics

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Motivation

Research Topic:

- Which model should be used to model dynamics of equity indices

- Capturing stylized facts in the data:
  - Non-normality
  - Heavy tails
  - Skewness
  - Volatility clustering
  - Leverage effect

- What has been done:
  - GBM: Black, Scholes, Merton Model (1973)
  - Jumps in returns: Merton (1976)
  - Stochastic volatility (SV): Heston (1993)
Objective of the Paper

Empirical findings:

1. Heston model is misspecified
2. Jump components in returns reduce misspecification

Two approaches to better capture stock return properties:

- Eraker et al. (2003): affine SV structure plus jumps in returns and volatility (based on Duffie, Pan, Singleton (2000))
- Christofferson et al. (2007): non-affine structure of SV process (extending Heston (1993))

Objective of the paper:

- compare the two approaches
- combine the two approaches
- consider SV, SVJ, and SVCJ model classes
- estimate parameters via Markov Chain Monte Carlo (MCMC)
- compare model performance (statistically / economically)
Objective of the paper / Research Questions:

1. Does the performance of non-affine SV models improve by including jumps (in general)?

2. Do we still have to leave the class of affine models after including jumps?
Agenda:

1. Motivation
2. Model Setup and Estimation
3. Data Set
4. Results
5. Conclusion
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**SVCJ model specification**

- We assume that the logarithm of the stock price solves

\[
\begin{align*}
    dY_t &= \mu dt + \sqrt{V_t} dW^y_t + \xi^y dN_t \\
    dV_t &= \kappa V_t^a (\theta - V_t) dt + V_t^b \sigma_V dW^v_t + \xi^v dN_t
\end{align*}
\]

- **Assumptions**
  - $dW^y_t$, $dW^v_t$ are Brownian motions with correlation $\rho$
  - $N_t$ is a Poisson process with intensity $\lambda$
  - SVCJ: $\xi^v_t \sim \text{Exp}(\mu_v)$; $\xi^v_t | \xi^v_{t-} \sim \mathcal{N}(\mu_y + \rho_j \xi^v_{t-}, \sigma_y)$
  - $a \in [0; 1]$ and $b \in [1/2; 1; 3/2]$

- Stochastic volatility (SV) and stochastic volatility with jumps in returns (SVJ) are special cases
Model Setup (cont’d)

Model specifications for each model class

\[ dV_t = \kappa V_t^a (\theta - V_t) dt + V_t^b \sigma_V dW_t^V \]

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>Name</th>
<th>Features</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.5</td>
<td>SQR</td>
<td>variance drift is linear in variance square root diffusion</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>SQRN</td>
<td>variance drift is nonlinear in variance square root diffusion</td>
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<tr>
<td>0.0</td>
<td>1.0</td>
<td>VAR</td>
<td>variance drift is linear in variance linear diffusion</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>VARN</td>
<td>variance drift is nonlinear in variance linear diffusion</td>
</tr>
<tr>
<td>0.0</td>
<td>1.5</td>
<td>3/2</td>
<td>variance drift is linear in variance 3/2 diffusion</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5</td>
<td>3/2N</td>
<td>variance drift is nonlinear in variance 3/2 diffusion</td>
</tr>
</tbody>
</table>

- Model classes: SV, SVJ, SVCJ (overall 18 different models)
Criteria of Model Fit

Statistical model choice is based on using

- **Quantile to quantile plots (QQ-Plots)**
  - plot quantiles of estimated errors of return equation against standard normal distribution
  
  $$
  \varepsilon_{t+1}^y = \left( R_{t+1} - \mu - \xi_{t+1}^y J_{t+1} \right) / V_t^b
  $$

- **Deviance Information Criterion (DIC)**
  - like any other information criterion combines a term for model fit and model complexity
  
  $$
  DIC = \bar{D} + p_D
  $$

- **Bayes Factors**
  - ratio of probabilities of two models given the data
  
  $$
  \frac{p(M_1|data)}{p(M_2|data)}
  $$
Agenda:
Data Set

Data used

- Time series data
  - daily returns of S&P500 index
  - daily returns of NASDAQ index

- Sample period from January 2, 1986 to July 31, 2008
  - robustness check: using different sub samples
  - including data up to first half of 2009 (Lehman)

- MCMC procedure
  - number of draws 500,000; burn-in period of 200,000
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Comparing models via QQ-Plots:

- SV-SQR
- SV-3/2N
- SVCJ-SQR

- Heston (SV-SQR) is misspecified
- Non-affine SV model, and affine jump diffusion model have a good fit
- Performance in the tails is much better
### Deviance Information Criterion

#### Comparing models via DIC-Statistics:

<table>
<thead>
<tr>
<th>Model</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 SVCJ-VARN</td>
<td>14023.51</td>
</tr>
<tr>
<td>2 SVJ-3/2N</td>
<td>14062.19</td>
</tr>
<tr>
<td>3 SVCJ-3/2N</td>
<td>14091.67</td>
</tr>
<tr>
<td>4 SVJ-VAR</td>
<td>14103.33</td>
</tr>
<tr>
<td>5 SVJ-VARN</td>
<td>14125.84</td>
</tr>
<tr>
<td>6 SVCJ-SQR</td>
<td>14144.14</td>
</tr>
<tr>
<td>7 SVCJ-VAR</td>
<td>14177.38</td>
</tr>
<tr>
<td>8 SVJ-SQR</td>
<td>14177.90</td>
</tr>
<tr>
<td>9 SV-3/2N</td>
<td>14199.65</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 SVJ-SQRN</td>
<td>14212.51</td>
</tr>
<tr>
<td>11 SVCJ-SQRN</td>
<td>14222.77</td>
</tr>
<tr>
<td>12 SV-VARN</td>
<td>14240.15</td>
</tr>
<tr>
<td>13 SV-VAR</td>
<td>14263.10</td>
</tr>
<tr>
<td>14 SV-SQR</td>
<td>14281.59</td>
</tr>
<tr>
<td>15 SV-SQRN</td>
<td>14401.61</td>
</tr>
<tr>
<td>16 SV-3/2</td>
<td>15355.37</td>
</tr>
<tr>
<td>17 SVCJ-3/2</td>
<td>15373.22</td>
</tr>
<tr>
<td>18 SVJ-3/2</td>
<td>15474.34</td>
</tr>
</tbody>
</table>

- **SV models are outperformed** by jump diffusion models (affine as well as non-affine)
- **Non-affine models perform best**
- **Of all models with linear drift** SVCJ-SQR is second best
Bayes Factors

Comparing models via Bayes Factors:

<table>
<thead>
<tr>
<th>(a; b)</th>
<th>SVJ vs. SV</th>
<th>SVCJ vs. SV</th>
<th>SVCJ vs. SVJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0;0.5)</td>
<td>19.74</td>
<td>4.57</td>
<td>-15.17</td>
</tr>
<tr>
<td>(1.0;0.5)</td>
<td>19.58</td>
<td>41.36</td>
<td>21.78</td>
</tr>
<tr>
<td>(0.0;1.0)</td>
<td>27.82</td>
<td>25.66</td>
<td>-2.15</td>
</tr>
<tr>
<td>(1.0;1.0)</td>
<td>25.05</td>
<td>26.50</td>
<td>1.45</td>
</tr>
<tr>
<td>(0.0;1.5)</td>
<td>40.46</td>
<td>38.70</td>
<td>-1.76</td>
</tr>
<tr>
<td>(1.0;1.5)</td>
<td>34.03</td>
<td>43.30</td>
<td>9.26</td>
</tr>
</tbody>
</table>

- Computation of Bayes factors only for nested models
- Ratio from 6 to 10: strong evidence for model in nominator
- **SV models are outperformed** by jump diffusion models (affine as well as non-affine)
- Mixed results for SVCJ vs. SVJ
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Contributions and Findings

**Contributions:**

- **Combine** two approaches to overcome model misspecification
- **Estimate** model parameters via MCMC
- **Compare** different model specifications by several test statistics
Findings:

**Overall result:**

1. **Jump models** are clearly preferred by test statistics
   - results hold for affine and non-affine specifications

2. **Non-affine** models exhibit a good fit to the data and are worth investigating
   - mathematical and economic properties are unknown

3. **Affine models** with jumps similar performance than non-affine models
   - we tend to prefer affine models since they are well understood (closed form solution, mathematical properties)
Thank you very much!
Estimation

Estimation is based on

- Euler discretization yields

\[
R_{t+1} = \mu + \sqrt{V_t} \varepsilon^y_{t+1} + \xi^y_{t+1} J_{t+1} \\
V_{t+1} = V_t + \kappa V_t (\theta - V_t) + \sigma_v V_t^b \varepsilon^v_{t+1} + \xi^v_{t+1} J_{t+1}.
\]

where

- \( R_{t+1} = Y_{t+1} - Y_t \)
- \( J_{t+1} = N_{t+1} - N_t \)
- \( \varepsilon^i_{t+1} = W^i_{t+1} - W^i_t \) for \( i = y, v \)

Aim is to estimate

- Parameters: \( \Theta = (\rho, \kappa, \theta, \sigma_v, \mu, \mu_y, \sigma_y, \lambda, \mu_v, \rho_j) \)
- Latent variables: \( X = \{ V_t, J_t, \xi^y_t, \xi^v_t \}_{t=1}^T \)
Bayesian Framework

Baysian framework used for estimation

- Posterior distribution (PD) is given by Bayes’ Theorem

\[ p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto p(y|\theta)p(\theta) \]

- PD combines information in model and prices
- likelihood \( p(y|\theta) \) is given by the model
- prior \( p(\theta) \) exogenously given (uninformative)

- As point estimator for parameters from posterior we use

\[ E(\theta|y) = \int \theta p(\theta|y) d\theta \]
Bayesian Framework

Problems for estimation procedure:

- Posterior distribution does not take the form of a well known density
  - no closed form solution
- Posterior distribution is high dimensional
- Simultaneous estimation of parameters and latent variables

Solution: Using Markov Chain Monte Carlo (MCMC)
**MCMC**

**MCMC in a nutshell:**
- We want to sample from PD $p(\theta|y)$

**MC:** Construct Markov Chain which converges to the PD
- Given initial values $\theta^{(0)}$ draw a sequence
  \[
  \theta_1^{(1)} \sim p(\theta_1|\text{all other parameters, } Y) \\
  \vdots \\
  \theta_K^{(1)} \sim p(\theta_K|\text{all other parameters, } Y)
  \]
- The resulting sequence $\{\theta^{(g)}\}_{g=1}^G$ converges to PD

**MC:** Calculate point estimators by approximating
\[
E(\theta|y) = \int \theta p(\theta|y) d\theta \approx \frac{1}{N} \sum_{n=G+1}^{N} \theta^{(n)}
\]
Comment on non-linear drift

**Problems with non-linear drift specification**

- Non-linear drift specification means $a = 1$
  
  \[
  dV_t = \kappa V_t^a(\theta - V_t)dt + V_t^b \sigma_v dW_t^v
  \]
  
  \[
  dV_t = \kappa V_t \theta dt - \kappa V_t^2 dt + V_t^b \sigma_v dW_t^v
  \]

- Drift and diffusion term vanishes when variance hits 0
  - In this case long run mean of variance is 0

- Is this specification economically questionable!?

- Further research needed
  (solution: condition process on not hitting 0)
Summerize Results:

1. In terms of QQ plots, affine jump diffusion models are similar to non-affine models.
2. Affine jump diffusion model is the second best DIC statistic of models with linear drift.
3. Jump diffusion models are clearly preferred by DIC statistic / Bayes factors.
4. We suggest further investigation of non-affine models, due to good statistical properties.
5. We tend to prefer affine model class:
   - performance of affine models similar as non-affine models
   - mathematical and statistical properties of affine models are well known.
Future Research:

- Comparison to Levy processes
- Using high frequency data
- Different drift specifications (regime switching models)
- Consider out-of-sample test
  - What about the conditional return distribution?
Criteria of Model Fit

**Economic model choice is based on using**

- Model capability of capturing the smile
  
  (a) calculation option prices via monte carlo (different strikes and maturities)

  (b) backing out the implied volatility

  (c) comparing model generated smiles with actual smile observed in the data (randomly chosen day)

  (d) consistent estimation of $Q$ and $P$ parameters by means of return and option data (Broadie et al. (2007))

\[
(\hat{\Theta}_Q, \hat{V}_t) = \arg\min \sum_{t,n} \left[ IV_t(K_n, \tau_n, S_t, r) - IV(V_t, \Theta^Q|\Theta^P, K_n, \tau_n, S_t, r) \right]^2
\]