Convertible Subordinated Debt Financing and Optimal Investment Timing

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Many companies issue convertible debt as a means of debt financing.

Convertible debt is an attractive security
– The seller may not incur the debt in conversion and can also enjoy interest tax shields relative to equity.
– The buyer has an option to convert into equity when its value is higher.

Firms issuing the convertible subordinated debt have been common.
1. Introduction (2)

- Senior-sub structure for equity, straight debt and convertible debt

- Senior-sub structure
  - The junior security holders will not get paid at all until the senior security holders are completely paid off at default. (Black and Cox(1976), Sundaresen and Wang(2006))
1. Introduction (3)

- The firm’s investment and financing decisions
  - Real options framework

- The investment problems for the firm
  - All-equity financing
    (McDonald and Siegel(1986), Dixit and Pindyck(1994))
  - Straight debt financing
1. Introduction (4)

- The investment problems for the firm
  - Convertible debt financing
    (Lyandres and Zhdanov(2006b), Yagi et al. (2008), Egami(2010))

  **Assumption:** Same priority (*Pari passu*)

- Our study: the senior-sub structure of equity, straight debt and convertible debt
The optimal investment policy for the firm financed by issuing equity, straight debt and convertible debt.
Our model:

- The investment is financed with equity, straight debt and convertible debt with senior-sub structure.
- Straight debt and convertible debt are issued at par.

Our objectives:

- Senior-sub structure
  - Optimal policies for financing and investment
  - Optimal capital structure
2. The Model (1)

- A firm with an option to invest at any time.

- \( I \): a fixed investment cost

- The firm finances the cost of investment with equity, straight debt and convertible debt.

<table>
<thead>
<tr>
<th>Equity</th>
<th>Straight debt with face value ( F_s ), coupon rate ( s )</th>
<th>Convertible debt with face value ( F_c ), coupon rate ( c )</th>
</tr>
</thead>
</table>
2. The Model (2)

- \( \tau \): a constant corporate tax rate
- Coupon payments are tax-deductible.
- Straight debt and convertible debt are issued at par.
2. The Model (3)

Suppose the firm observes the demand shock \( X_t \) for its product

- \( X_t \): a geometric Brownian motion

\[
dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x
\]  
\[\text{(1)}\]

- \( \mu \): the risk-adjusted expected growth rate
- \( \sigma \): the volatility
- \( W_t \): a standard Brownian motion

Once the investment option is exercised, the firm can receive the instantaneous profit

\[
\pi(X_t, s, c) = (1 - \lambda)(QX_t - sF_s - cF_c)
\]  
\[\text{(2)}\]

- \( Q \): the quality produced from the asset in place
The investment is financed with equity, straight debt and convertible debt.

1. Optimal investment policy

2. The values of equity, straight debt and convertible debt with same priority

3. The values of equity, straight debt and convertible debt with senior-sub structure

4. Optimal capital structure
2.1. Optimal Investment Policy (1)

- The equity holders of the firm which invests
  - to select the optimal investment timing, observing the demand shock $X_t$

- $V(x)$: The value of firm which is financed with equity, straight debt and convertible debt

\[
V(x) = E(x) + D_s(x) + D_c(x)
\]  

- $E(x)$: the value of equity
- $D_s(x)$: the value of straight debt
- $D_c(x)$: the value of convertible debt
2.1. Optimal Investment Policy (2)

- The value of investment partially financed with straight debt and convertible debt

\[ F(x) = \sup_{T \in T_0,\infty} E^x_0 \left[ e^{-rT} (E(X_T) - (I - D_s(X_T) - D_c(X_T))) \right] \]

\[ = \sup_{T \in T_0,\infty} E^x_0 \left[ e^{-rT} (V(X_T) - I) \right] \] (4)

- \( x^* \) : the optimal investment threshold

- Since straight debt and convertible debt are issued at par, the coupon rates \( s \) and \( c \) are determined such that

\[ D_s(x^*) = F_s \] (5)

\[ D_c(x^*) = F_c \] (6)
2.2. Same priority (1)

- The values of equity, straight debt and convertible debt with the same priority

- The convertible debt holders can convert the debt into a fraction $\eta$ of the original equity;

$$\eta = \alpha c F_c$$  \hspace{1cm} (7)

- $\alpha$ : constant

- The investment option has been exercised.

- From the issue of debt, the optimal default policy is established.
2.2. Same priority (2)

- The optimal default policy of the equity holders
  - to select the default time $T_d$, maximizing the equity value

- The optimal conversion policy of the convertible debt holders
  - to select the conversion time $T_c$, maximizing the value of convertible debt

- The optimal problems for the holders of equity and convertible debt must be solved simultaneously.
2.2. Same priority (3)

- At default time $T_d$
  - Equity holders cannot receive anything.
  
  - Straight debt holders receive
    \[ D_s(X_{T_d}) = \frac{F_s}{F_s + F_c} (1 - \theta)\epsilon(X_{T_d}) \] (8)
  
  - Convertible debt holders receive
    \[ D_c(X_{T_d}) = \frac{F_c}{F_s + F_c} (1 - \theta)\epsilon(X_{T_d}) \] (9)

\[ \theta \quad : \text{the proportional bankruptcy cost} \]
\[ 0 \leq \theta \leq 1 \]
2.2. Same priority (4)

The value of equity at investment time $t$

$$E(x) = \sup_{T_d \in \mathcal{T}_{t,\infty}} E_t^x \left[ \int_t^{T_c^* \wedge T_d} e^{-r(u-t)}(1 - \tau)(Q X_u - s F_s - c F_c) du + 1_{\{T_c^* < T_d\}} e^{-r(T_c^* - t)} \frac{1}{1 + \eta} E_a(X_{T_c^*}) \right]$$

- By converting, the equity value is diluted.
  - The dilution factor: $$\frac{1}{1 + \eta}$$
  - $E_a(x)$: the value of equity after conversion
2.2. Same priority (5)

- The value of convertible debt at investment time $t$

$$D_c(x) = \sup_{T_c \in \mathcal{T}_{t, \infty}} E_t^x \left[ \int_t^{T_c \wedge T_d^*} e^{-r(u-t)} cF_c du 
+ 1_{\{T_c < T_d^*\}} e^{-r(T_c-t)} \frac{\eta}{1 + \eta} E_a(X_{T_c}) 
+ 1_{\{T_d^* < T_c\}} e^{-r(T_d^*-t)} \frac{F_c}{F_s + F_c} (1 - \theta) \epsilon(X_{T_d^*}) \right]$$ \hspace{1cm} (11)

- Optimal default time

$$T_d^* = \inf\{T_d \in [t, \infty) \mid X_{T_d} \leq x_d\}$$ \hspace{1cm} (12)

- Optimal conversion time

$$T_c^* = \inf\{T_c \in [t, \infty) \mid X_{T_c} \geq x_c\}$$ \hspace{1cm} (13)

- $x_c$ : Optimal conversion threshold
2.2. Same priority (6)

- The value of straight debt at investment time \( t \)

\[
D_s(x) = E_t^x \left[ \int_t^{T_c^* \wedge T_d^*} e^{-r(u-t)} s F_s d\mu + 1_{\{T_c^* < T_d^*\}} e^{-r(T_c^* - t)} D_{s,a}(X_{T_c^*}) \right. \\
+ 1_{\{T_d^* < T_c^*\}} e^{-r(T_d^* - t)} \frac{F_s}{F_s + F_c} (1 - \theta) \epsilon(X_{T_d^*}) \]

(14)

- \( D_{s,a}(x) \): the value of straight debt after conversion
2.2. Same priority (7)

- The post-conversion value of equity

\[ E_a(x) = e(x) - \frac{(1 - \tau)sF_s}{r} - \left( \frac{x}{x_{d,c}} \right)^{\beta_2} \left( e(x_{d,c}) - \frac{(1 - \tau)sF_s}{r} \right) \]  

(15)

- The post-conversion value of straight debt

\[ D_{s,a}(x) = \frac{sF_s}{r} - \left( \frac{x}{x_{d,c}} \right)^{\beta_2} \left( \frac{sF_s}{r} - (1 - \theta)e(x_{d,c}) \right) \]  

(16)

\[ -\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}} < 0 \]

- The post-conversion default threshold

\[ x_{d,c} = \frac{\beta_2}{\beta_2 - 1} \frac{r - \mu sF_s}{r} Q \]  

(17)
2.3. Senior-Sub Structure (1)

- The values of equity, straight debt and convertible debt with the senior-sub structure

- At default time $T_d$
  - Straight debt holders receive
    \[ D_s(X_{T_d}) = \min(F_s, (1 - \theta)\epsilon(X_{T_d})) \]  
  \[ (18) \]
  - Convertible debt holders receive
    \[ D_c(X_{T_d}) = \min(F_c, \max((1 - \theta)\epsilon(X_{T_d}) - F_s, 0)) \]  
  \[ (19) \]
  - Equity holders receive
    \[ E(X_{T_d}) = \max((1 - \theta)\epsilon(X_{T_d}) - F_s - F_c, 0) \]  
  \[ (20) \]
2.3. Senior-Sub Structure (2)

- The value of equity at investment time $t$

$$E(x) = \sup_{T_d \in \mathcal{T}_t,\infty} E^x_t \left[ \int_t^{T_c^* \wedge T_d} e^{-r(u-t)} (1 - \tau)(Q X_u - s F_s - c F_c) du ight. $$

$$+ 1\{T_c^* < T_d\} e^{-r(T_c^*-t)} \frac{1}{1 + \eta} E_a(X_{T_c^*}) $$

$$+ 1\{T_d < T_c^*\} e^{-r(T_d-t)} \max((1 - \theta)\epsilon(X_{T_d}) - F_s - F_c, 0) \right]$$ (21)

- The value of convertible debt at investment time $t$

$$D_c(x) = \sup_{T_c \in \mathcal{T}_t,\infty} E^x_t \left[ \int_t^{T_c \wedge T_c^*} e^{-r(u-t)} c F_c du ight.$$

$$+ 1\{T_c < T_c^*\} e^{-r(T_c-t)} \frac{\eta}{1 + \eta} E_a(X_{T_c}) $$

$$+ 1\{T_c^* < T_c\} e^{-r(T_c^*-t)} \min(F_c, \max((1 - \theta)\epsilon(X_{T_c^*}) - F_s, 0)) \right]$$ (22)
2.3. Senior-Sub Structure (3)

- The value of straight debt at investment time $t$

\[
D_s(x) = E_t^x \left[ \int_t^{T_c^* \wedge T_d^*} e^{-r(u-t)} s F_s du + 1_{\{T_c^* < T_d^*\}} e^{-r(T_c^*-t)} D_{s,a}(X_{T_c^*}) + 1_{\{T_d^* < T_c^*\}} e^{-r(T_d^*-t)} \min(F_s, (1 - \theta)\epsilon(X_{T_d^*})) \right] \tag{23}
\]

- At post-conversion default time $T_{d,c}$
  - Straight debt holders receive

\[
D_{s,a}(X_{T_{d,c}}) = \min(F_s, (1 - \theta)\epsilon(X_{T_{d,c}})) \tag{24}
\]
  - Equity holders receive

\[
E(X_{T_{d,c}}) = \max((1 - \theta)\epsilon(X_{T_{d,c}}) - F_s, 0) \tag{25}
\]
2.3. Senior-Sub Structure (4)

- The post-conversion value of equity

\[ E_a(x) = \sup_{T_{d,c} \in T_{t,\infty}} E_t^x \left[ \int_t^{T_{d,c}} e^{-r(u-t)} (1 - \tau)(Q X_u - s F_s) du + e^{-r(T_{d,c}-t)} \max((1 - \theta)\epsilon(X_{T_{d,c}}) - F_s, 0) \right] \]

- Optimal post-conversion default threshold

\[ T_{d,c}^* = \inf \{ T_{d,c} \in [t, \infty) \mid X_{T_{d,c}} \leq x_{d,c} \} \]

- The post-conversion value of convertible debt

\[ D_{s,a}(x) = E_t^x \left[ \int_t^{T_{d,c}^*} e^{-r(u-t)} s F_s du + e^{-r(T_{d,c}^*-t)} \min(F_s, (1-\theta)\epsilon(X_{T_{d,c}^*})) \right] \]
2.4. Optimal Capital Structure

- The optimal capital structure of the firm issuing equity, straight debt and convertible debt

- The face value of convertible debt $F_c$

  $$\eta = \alpha c F_c$$  \hspace{1cm} (7)

- The face value of straight debt $F_s$

  $$F(x) = \sup_{T \in T_0, \infty, F_s > 0} E_0^x \left[ e^{-rT} (V(X_T) - I) \right]$$  \hspace{1cm} (5)'
3. Numerical Analysis

- Optimal face values for straight debt and convertible debt
- Coupon rates for straight debt and convertible debt
- Threshold for investment
- Optimal leverage ratio

solving nonlinear simultaneous equations.
## Tab.1 Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity for $X_t$</td>
<td>$Q = 1$</td>
</tr>
<tr>
<td>Initial value of $X_t$</td>
<td>$x = 0.5$</td>
</tr>
<tr>
<td>Expected growth rate</td>
<td>$\mu = 0$</td>
</tr>
<tr>
<td>Volatility</td>
<td>$\sigma = 0.2$</td>
</tr>
<tr>
<td>Interest rate</td>
<td>$r = 0.05$</td>
</tr>
<tr>
<td>Investment cost</td>
<td>$I = 20$</td>
</tr>
<tr>
<td>Proportional constant on conversion</td>
<td>$\alpha = 1.5$</td>
</tr>
<tr>
<td>Proportional bankruptcy cost</td>
<td>$\theta = 0.3$</td>
</tr>
<tr>
<td>Corporate tax rate</td>
<td>$\tau = 0.3$</td>
</tr>
</tbody>
</table>
Tab. 2 Optimal capital structure for the conversion ratio

<table>
<thead>
<tr>
<th></th>
<th>Senior-sub structure</th>
<th>Same priority</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$F_c$</td>
<td>6.6</td>
<td>11.7</td>
</tr>
<tr>
<td>$F_s$</td>
<td>20.6</td>
<td>16.7</td>
</tr>
<tr>
<td>$c$</td>
<td>0.051</td>
<td>0.057</td>
</tr>
<tr>
<td>$s$</td>
<td>0.064</td>
<td>0.061</td>
</tr>
<tr>
<td>$x^*$</td>
<td>2.320</td>
<td>2.355</td>
</tr>
<tr>
<td>$\frac{D_c(x^<em>) + D_s(x^</em>)}{V(x^*)}$</td>
<td>0.727</td>
<td>0.760</td>
</tr>
</tbody>
</table>
### Tab. 2.1 Optimal capital structure for the conversion ratio

<table>
<thead>
<tr>
<th></th>
<th>Senior-sub structure</th>
<th>Same priority</th>
<th>Increase -decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>$F_c$</td>
<td>6.6</td>
<td>11.7</td>
<td>16.3</td>
</tr>
<tr>
<td>$F_s^*$</td>
<td>20.6</td>
<td>16.7</td>
<td>12.9</td>
</tr>
<tr>
<td>$c$</td>
<td>0.051</td>
<td>0.057</td>
<td>0.061</td>
</tr>
<tr>
<td>$s$</td>
<td>0.064</td>
<td>0.061</td>
<td>0.057</td>
</tr>
</tbody>
</table>

The conversion ratio $\eta$ increases.

Face value $F_c$ and coupon rate $c$ for convertible debt increase.

-Convertible debt holders can receive more equity in conversion.
Tab.2.1 Optimal capital structure for the conversion ratio

<table>
<thead>
<tr>
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<th>Senior-sub structure</th>
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<tr>
<td>$\eta$</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>$F^*_c$</td>
<td>6.6</td>
<td>11.7</td>
<td>16.3</td>
</tr>
<tr>
<td>$F^*_s$</td>
<td>20.6</td>
<td>16.7</td>
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</tr>
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<td>$c$</td>
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<td>0.057</td>
</tr>
</tbody>
</table>

The conversion ratio $\eta$ increases.

Face value $F^*_s$ and coupon rate $s$ for straight debt decrease.

- Balance between convertible debt and straight debt
Tab.2.2 Optimal capital structure for the conversion ratio

<table>
<thead>
<tr>
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<th>Senior-sub structure</th>
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<th>Increase-decrease</th>
</tr>
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<tbody>
<tr>
<td>$\eta$</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>$x^*$</td>
<td>2.320</td>
<td>2.355</td>
<td>2.382</td>
</tr>
<tr>
<td>$\frac{D_c(x^<em>)+D_s(x^</em>)}{V(x^*)}$</td>
<td>0.727</td>
<td>0.760</td>
<td>0.780</td>
</tr>
</tbody>
</table>

The conversion ratio $\eta$ increases.

Investment threshold $x^*$ and leverage ratio $\frac{D_c(x^*)+D_s(x^*)}{V(x^*)}$ increase.

- Dilution for equity value becomes larger.
Tab.2.3 Optimal capital structure for the conversion ratio

<table>
<thead>
<tr>
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<th>Comparison</th>
<th>Same priority</th>
</tr>
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<tbody>
<tr>
<td>$\eta$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
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<td></td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>$F_c$</td>
<td>6.6</td>
<td>&lt;</td>
<td>7.3</td>
</tr>
<tr>
<td></td>
<td>11.7</td>
<td></td>
<td>13.2</td>
</tr>
<tr>
<td></td>
<td>16.3</td>
<td></td>
<td>18.4</td>
</tr>
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<td>$F_s^*$</td>
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<td>$s$</td>
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Face value for convertible debt is smaller.
Coupon rate for convertible debt is larger.
- Possibility that convertible debt holders cannot receive anything at default
### Tab.2.3 Optimal capital structure for the conversion ratio

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<tr>
<td>( s )</td>
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<td>0.061</td>
<td>0.057</td>
</tr>
</tbody>
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Face value for straight debt is larger.
Coupon rate for straight debt is smaller.
- Possibility that payoff to straight debt holders at default is higher.
Tab.2.4 Optimal capital structure for the conversion ratio

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Investment threshold and leverage ratio are higher.
- Possibility that convertible debt holders cannot receive anything at default
- Conversion occurs earlier.
- Equity value decreases from dilution.
4. Summary

- The optimal investment policy of the firm financed by issuing equity, straight debt and convertible debt
  - **Senior-sub structure**
  - **Straight debt and convertible debt are issued at par.**

- The senior-sub structure
  - Convertible debt
    - Face value is larger and coupon rate is smaller.
  - Straight debt
    - Face value is smaller and coupon rate is larger.
  - Investment occurs later.
  - Leverage ratio is higher.
Thank you.

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