State-Price Densities in the Commodity Market and Its Relevant Economic Implications

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(Incomplete and all comments are welcome.)
Motivation

“From a pricing perspective, SPDs are "sufficient statistics" in an economic sense - they summarize all relevant information about preferences and business conditions for purposes of pricing financial securities.”

--- Aït-Sahalia and Lo (1998, JF)

• SPDs reflect the beliefs of investors about the likelihood of possible states and their preferences towards these states.
• SPDs have important implications for derivative pricing.
• SPDs provide information about how the commodity market is segmented from other financial asset markets.
Motivation

Commodity derivatives market grows very fast: OTC commodity derivatives contracts is $6.4 trillion in 2006, about 14 times the value in 1998. (Bank for International Settlements, 2006)

Crude oil futures price and number of options written on 3-month contract.
Agenda

• Introduction
  • Concept Definition
  • Main Findings
  • Literature Review
• Methodology
  • Econometric Methodology
• Data
• Empirical Results
• Discussion
Concept Definition

State Price Densities (SPDs) $\xi$ is defined as

$$W_0 = E[\xi_T W_T]$$

(Arrow-Debreu price of per unit of probability.)
Concept Definition

State Price Densities (SPDs) $\xi$ is defined as

$$W_0 = E[\xi_T W_T]$$  \hspace{1cm} (1)

(Arrow-Debreu price of per unit of probability.)

A call option can therefore be priced as:

$$C(F_t, K, t, T) = E[\xi_T (F_T - K)^+ | \mathcal{F}_t]$$
$$= \int_{K}^{\infty} \xi_T(x)(F_T - K)P(F_T = x | \mathcal{F}_t)dx$$  \hspace{1cm} (2)
Concept Definition

For a call option,

\[ C(F_t, K, t, T) = E[\xi_T(F_T - K)^+ | \mathcal{F}_t] \]

\[ = \int_K^{\infty} \xi_T(x)(F_T - K)P(F_T = x | \mathcal{F}_t)dx \quad (2) \]

\[ C(F_t, K, t, T) = e^{-r(T-t)}E^Q[(F_T - K)^+] \]

\[ = e^{-r(T-t)} \int_K^{\infty} (F_T - K)P^Q(F_T = x | \mathcal{F}_t)dx \quad (3) \]
Concept Definition

For a call option,

\[
C(F_t, K, t, T) = E[\xi_T (F_T - K)^+ | \mathcal{F}_t] = \int_K^{\infty} \xi_T(x)(F_T - K)P(F_T = x | \mathcal{F}_t) dx
\]

(2)

\[
C'(F_t, K, t, T) = e^{-r(T-t)} E^Q[(F_T - K)^+]
\]

(3)

\[
= e^{-r(T-t)} \int_K^{\infty} (F_T - K)P^Q(F_T = x | \mathcal{F}_t) dx
\]

Projection of \( \xi \) onto the crude oil futures market is

\[
\tilde{\xi}(F_T; \mathcal{F}_t) = e^{-r(T-t)} \frac{P^Q(F_T | \mathcal{F}_t)}{P(F_T | \mathcal{F}_t)}
\]

(4)
Concept Definition

For a call option,

\[
\begin{align*}
C(F_t, K, t, T) &= E[\xi_T(F_T - K)^+ | \mathcal{F}_t] \\
&= \int_K^{\infty} \xi_T(x)(F_T - K) P(F_T = x | \mathcal{F}_t) dx \\
&= e^{-r(T-t)} E^Q[(F_T - K)^+] \\
&= e^{-r(T-t)} \int_K^{\infty} (F_T - K) P^Q(F_T = x | \mathcal{F}_t) dx
\end{align*}
\] (2)

Projection of \( \xi \) onto the crude oil futures market is

\[
\hat{\xi}(F_T; \mathcal{F}_t) = e^{-r(T-t)} \frac{P^Q(F_T | \mathcal{F}_t)}{P(F_T | \mathcal{F}_t)}
\] (4)

\[
P^Q(F_T | \mathcal{F}_t) = e^{r(T-t)} \frac{\partial^2 C(F_t, K, t, T)}{\partial K^2} |_{K=F_T}
\] (5)
Concept Definition

For a call option,

\[ C(F_t, K, t, T) = E[\xi_T(F_T - K)^+ | \mathcal{F}_t] \]

\[ = \int_K^\infty \xi_T(x)(F_T - K) P(F_T = x | \mathcal{F}_t) dx \]  \hspace{1cm} (2)

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Projection of \( \xi \) onto the crude oil futures market is

\[ \hat{\xi}(F_T; \mathcal{F}_t) = e^{-r(T-t)} \frac{P^Q(F_T | \mathcal{F}_t)}{P(F_T | \mathcal{F}_t)} \]  \hspace{1cm} (4)

\[ P^Q(F_T | \mathcal{F}_t) = e^{r(T-t)} \frac{\partial^2 C(F_t, K, t, T)}{\partial K^2} |_{K=F_T} \]  \hspace{1cm} (5)
Main Findings

Risk Neutral Densities

• Strongly deviate from the normal distribution;
• Skewness could be *either negative or positive* depending on maturity, slope and volatility;
• Distributions tend to be negatively skewed for short maturity contracts and positively skewed for long maturity contracts.

State Price Densities

• U-Shape SPDs. Investors assign high values for states with very high or low returns.
• Shape of SPDs varies with volatilities, implying the importance of incorporating stochastic volatility into options pricing.
Literature Review

**Equity Index Options**
- **SPDs:** Rosenberg and Engle (2002, JF), Bakshi, Madan and Panayotov (2009, JFE)

**Interest Rate Derivatives**
- **RNDs:** Beber and Brandt (2006, JME)
- **RNDs and SPDs:** Li and Zhao (2009, RFS)

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- **RNDs:** Melick and Thomas (1997, JFQA)
- **RNDs and SPDs:** Pan (Soon!)
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Econometric Methodology

1. For a given level of conditional variables $v=\{\text{slope, volatility}\}$, I collect those observations of call options on different dates whose conditional variables are within a window:

$$W(v,h) = \{1 \leq i \leq n, v_i \in [v-h^*, v+h^*]\}.$$  
(The optimal window size $h^*$ is chosen by iteration)

2. I filter the data according to slope and convexity constraints.

$$\min_m \sum_i (m_i - c_i)^2,$$

s.t.

$$0 \geq \frac{c_{i+1} - c_i}{x_{i+1} - x_i} \geq -1, \quad i = 1, \ldots, n - 1$$

$$\frac{c_{i+1} - c_i}{x_{i+1} - x_i} \geq \frac{c_i - c_{i-1}}{x_i - x_{i-1}}, \quad i = 2, \ldots, n - 1$$
Econometric Methodology

3. With the filtered data, I use the locally linear approach to estimate risk neutral densities (Aït-Sahalia and Duarte, 2003).

\[
\min_{\beta_0, \beta_1} \sum_{i=1}^{n} \left\{ m_i - \beta_0(x, v) - \beta_1(x, v) \times (x - x_i) \right\}^2 K_h(x_i - x, v_i - v)
\]

where \( K_h(x_i - x, v_i - v) \) is a joint kernel function and \( h \) is the bandwidth.

\[
\frac{\partial^2 \hat{m}(x, v)}{\partial x^2} = \frac{\partial \beta_1(x, v)}{\partial x}.
\]

We choose the optimal bandwidth by:

\[
\min_h \int_{h_x(v)} \min_{h_y(v)} E^Q[\hat{m}_h(x, v) - \hat{m}_y(x, v)]^2 dP(v)
\]

4. I get the physical densities using the standard kernel method.
Data

• Crude oil futures and options from NYMEX
  – Jan 02, 1990 – Dec 03, 2008
• Focus on 4 maturities: 3mn, 6mn, 12mn and 24mn.
• Choose call options data according to:
  – Jan 02, 1990 – Dec 14, 2006, keep data with open interest > 100 and price > $0.01;
  – Dec 15, 2006 – Dec 03, 2008, keep data with price > $0.01 (no open interest available).
• Conditional factors are calculated from futures:
  – $\text{Slope}(t) = \log[P(t;6mn)/P(t;3mn)];$ (Kogan, Livdan and Yaron, 2009, JF)
  – Volatility is from a leverage effect GARCH model for each maturity of futures;
  – All factors are adjusted to have a uniform distribution on $[0, 1]$. 
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SPD in the Commodity Market
Time series of futures price and conditional volatility

Density of conditional volatility (uniformed)
Empirical Results

• **Risk Neutral Densities** for 3mn, 6mn, 12mn and 24mn contract conditional on slope and volatility factors

• **Physical Densities** for 3mn, 6mn, 12mn and 24mn contract conditional on slope and volatility factors

• **State Price Densities** for 3mn, 6mn, 12mn and 24mn contract conditional on slope and volatility factors
3 Month

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Discussion

• Estimated SPDs are not smooth for many cases, especially for some short maturity contracts;
• Comparison with parametric method.

• Compare the risk neutral densities and SPDs during 2007-2008 and other periods.
• Infer investors’ expectation and preference by examining the time evolution of risk neutral densities (dynamics of mean and variance), especially for the period of 2007-2008.

Other economic implications to explore?