A Delay Financial Model with Stochastic Volatility; Martingale Method

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We extend a delayed geometric Brownian model by adding the stochastic volatility which is assumed to have fast mean reversion. By the martingale approach and singular perturbation method, we develop a theory for option pricing under this extended model.

Keywords: Black-Scholes, delay, stochastic volatility, martingale, option pricing, asymptotics.
The assumptions of Black-Scholes model for equity market

- It is possible to borrow and lend cash at a known constant risk-free interest rate
- The price follows a geometric Brownian motion with constant drift and volatility
- There are no transaction costs
- The stock does not pay a dividend
- All securities are perfectly divisible (i.e. it is possible to buy any fraction of a share)
- There are no restrictions on short selling
Causes of volatility smile/skew

- Crash protection/ Fear of crashes
- Transactions costs
- Local volatility
- Leverage effect
- CEV models
- Stochastic volatility
- Jumps/crashes
"Chartists believe that future prices depend on past movement of the asset price and attempt to forecast future price levels based on past patterns of price dynamics."

1. Contagion effects in a chartist-fundamentalist model with time delays

2. Speculative dynamics in a time-delay model of asset prices

"The insider knows that both the drift and the volatility of the stock price process are influenced by certain events that happened before the trading period started"

1. A stochastic delay financial model
   - George Stoica, American Mathematical Society, 133, 1837-1841, 2004
DSV model

\[ dX_t = \mu \xi(X_{t-a})X_t dt + f(Y_t)\eta(X_{t-b})X_t dW_t \]
\[ X_t = \psi(t), \quad t \in [-L, 0], \quad \text{where } L = \max \{a, b\} \]
\[ dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t, \]

where \( f(y) \) is a sufficiently smooth function, \( \xi \) is an arbitrary function, \( \eta \) is an arbitrary non-zero function, and \( \hat{Z}_t \) is a Brownian motion correlated with \( W_t \) such that \( d\hat{Z}_t = \rho dW_t + \sqrt{1 - \rho^2} dZ_t \), where \( W_t \) are \( Z_t \) are independent Brownian motions.
The path of DSV model

\[ \xi(x) = x^{0.2}, \quad \eta(x) = x^{0.001} \]
Theorem 1

There exists a unique and positive solution for the DSV equation on $t \in [0, T]$ by $(k + 1)$-step computations as follows:

$$X_t = X_{k \ell} \exp \left( \mu \int_{k \ell}^{t} \xi(X_s-a) - \frac{1}{2} f^2(Y_s) \eta^2(X_{s-b}) ds \right. $$

$$+ \left. \int_{k \ell}^{t} \eta(X_{s-b}) dW_s \right)$$

for the positive integer $k$ with $t \in [k \ell \wedge T, (k + 1) \ell \wedge T]$ where $\ell = \min\{a, b\}$. 
By Girsanov theorem, we can guarantee the existence of an equivalent martingale measure. Define $W_t^*$ and $Z_t^*$ as follows:

$$W_t^* = W_t + \int_0^t \frac{\mu \xi(X_s-a) - r}{f(Y_s)\eta(X_s-b)} ds$$

$$Z_t^* = Z_t + \int_0^t \gamma_s ds$$

where $\gamma_t$ is an adapted process to be determined.
By Girsanov theorem, we have an equivalent martingale measure \( Q \) given by the Radon-Nikodym derivative

\[
\frac{dQ}{dP} = \exp \left( -\frac{1}{2} \int_0^t \left[ \left( \frac{\mu \xi(X_{s-a}) - r}{f(Y_s)\eta(X_{s-b})} \right)^2 + \gamma_s^2 \right] ds \right. \\
- \int_0^t \frac{\mu \xi(X_{s-a}) - r}{f(Y_s)\eta(X_{s-b})} dW_s - \int_0^t \gamma_s dZ_s \right)
\]
**DSV model**

**SDDE under equivalent martingale measure** $Q$

**Model**

**Under $Q$**

\[
dX_t = r\xi(X_{t-a})X_t dt + f(Y_t)\eta(X_{t-b})X_t dW^*_t
\]

\[
X_t = \psi(t), \quad t \in [-L, 0], \quad \text{where} \quad L = \max\{a, b\}
\]

\[
dY_t = \left[\alpha(m - Y_t) - \beta \Lambda(Y_t, X_{t-a}, X_{t-b}, X_t)\right] dt
\]

\[
+ \beta d\widehat{Z}^*_t
\]

where

\[
\widehat{Z}^*_t = \rho W_t^* + \sqrt{1 - \rho^2} Z_t^*
\]

\[
\Lambda(Y_t, X_{t-a}, X_{t-b}, X_t) = \rho \frac{\mu \xi(X_{t-a}) - r}{f(Y_t)\eta(X_{t-b})} + \sqrt{1 - \rho^2} \gamma_t
\]
Now, we assume to have fast mean reversion. So, we introduce $\varepsilon$ as the inverse of the rate of mean reversion $\alpha$:

$$
\varepsilon = \frac{1}{\alpha}
$$

And, the long-run distribution of the OU process $Y_t$ is assumed to have the moderate variance $\nu^2 = \frac{\beta^2}{2\alpha} = O(1)$ so that

$$
\beta = \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}}
$$
In term of the small parameter $\varepsilon$, the DSV model becomes

**The DSV model**

\[
\begin{align*}
    dX_t^\varepsilon &= r\xi(X_{t-a}^\varepsilon)X_t^\varepsilon \, dt + f(Y_t^\varepsilon)\eta(X_{t-b}^\varepsilon)X_t^\varepsilon \, dW_t^* \\
    X_t^\varepsilon &= \psi(t), \quad t \in [-L, 0], \text{ where } L = \{a, b\} \\
    dY_t^\varepsilon &= \left(\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t^\varepsilon, X_{t-a}^\varepsilon, X_{t-b}^\varepsilon, X_t^\varepsilon)\right) \, dt \\
    &\quad + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}_t^* \\
\end{align*}
\]

where $\Lambda(Y_t^\varepsilon, X_{t-a}^\varepsilon, X_{t-b}^\varepsilon, X_t^\varepsilon) = \rho \frac{\mu\xi(X_{t-a}^\varepsilon) - r}{f(Y_t^\varepsilon)\eta(X_{t-b}^\varepsilon)} + \sqrt{1 - \rho^2} \gamma_t$
Suppose no-arbitrage opportunity. The option price $P^\varepsilon(t)$ at time $t$ of a derivative with terminal payoff function $h$ is given by

$$P^\varepsilon(t) = \mathbb{E}^* \{ e^{-r(T-t)} h(X^\varepsilon_T) | \mathcal{F}_t \}$$

where the conditional expectation is taken under the equivalent martingale measure $Q$, and $\mathcal{F}_t$ is a filtration with respect to the past of $(X^\varepsilon_t, Y^\varepsilon_t)$. 
Goal: Approximation

To find \( Q^\varepsilon \) such that \( P^\varepsilon(t) = Q^\varepsilon(t, X^\varepsilon_t) + O(\varepsilon) \)
The discounted price $M_t^\varepsilon$ defined by

$$M_t^\varepsilon = e^{-rt} P^\varepsilon(t) = \mathbb{E}^* \{ e^{-rT} h(X_T^\varepsilon) | \mathcal{F}_t \}$$

is martingale with a terminal value given by

$$M_T^\varepsilon = e^{-rT} h(X_T^\varepsilon)$$
A motivated theorem for finding $Q^\varepsilon(t, X_t)$

**Theorem 2**

Let $Q^\varepsilon(t, x)$ be a two-variable function with the following conditions:

(i) $Q^\varepsilon(t, x)$ satisfies $Q^\varepsilon(T, x) = h(x)$ at the final time $T$

(ii) $e^{-rt}Q^\varepsilon(t, X^\varepsilon_t)$ can be decomposed as

$$e^{-rt}Q^\varepsilon(t, X^\varepsilon_t) = \tilde{M}^\varepsilon_t + R^\varepsilon_t$$

where $\tilde{M}^\varepsilon$ is a martingale and $R^\varepsilon_t$ is of order $\varepsilon$

Then $P^\varepsilon(t) = Q^\varepsilon(t, X^\varepsilon_t) + O(\varepsilon)$
Theorem 2, continued

Proof

Let \( N_t^\varepsilon = e^{-rt} Q^\varepsilon(t, X_t^\varepsilon) \)

Then, from the condition (i) and (ii),

\[
M_T^\varepsilon = N_T^\varepsilon \text{ and } N_t^\varepsilon = \tilde{M}_t^\varepsilon + R_t^\varepsilon
\]

By taking a conditional expectation with respect to \( \mathcal{F}_t \) on both sides of the equality \( N_t^\varepsilon = \tilde{M}_t^\varepsilon + R_t^\varepsilon \), we have

\[
M_t^\varepsilon = \mathbb{E}^* \{ M_T^\varepsilon | \mathcal{F}_t \} = \mathbb{E}^* \{ N_T^\varepsilon | \mathcal{F}_t \}
\]
proof, continued

\[
\begin{align*}
&= \mathbb{E}^* \{ \tilde{M}_t^\varepsilon + R_t^\varepsilon | \mathcal{F}_t \} \\
&= \tilde{M}_t^\varepsilon + \mathbb{E}^* \{ R_T^\varepsilon | \mathcal{F}_t \} \\
&= N_t^\varepsilon + \mathbb{E}^* \{ R_T^\varepsilon | \mathcal{F}_t \} - R_t^\varepsilon \\
&= N_t^\varepsilon + \mathcal{O}(\varepsilon)
\end{align*}
\]

Therefore, by multiplying $e^{rt}$ on both sides, we obtain

\[
P^\varepsilon(t) = Q^\varepsilon(t, X_t^\varepsilon) + \mathcal{O}(\varepsilon)
\]
Now, we assume that one can choose $\gamma_t$ such that $\Lambda$ in the DSV model becomes a function depending upon only of $Y^\varepsilon_t$. That is,

$$dY^\varepsilon_t = \left( \frac{1}{\varepsilon}(m - Y^\varepsilon_t) - \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y^\varepsilon_t) \right) dt + \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}^*_t$$

Then we obtain the infinitesimal generator $\varepsilon^{-1} \mathcal{L}^\varepsilon_Y$ of $Y^\varepsilon_t$ where

$$\mathcal{L}^\varepsilon_Y = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y - \nu \sqrt{2\varepsilon}\Lambda(y)) \frac{\partial}{\partial y}$$
Assume that $\Lambda(y)$ is bounded. Then $Y^\varepsilon$ has a unique invariant distribution given by the probability density $\Phi_\varepsilon$:

$$\Phi_\varepsilon(y) = J_\varepsilon \exp \left( - \frac{(y - m)^2}{2\nu^2} - \frac{\sqrt{2\varepsilon}}{\nu} \tilde{\Lambda}(y) \right)$$

where $\tilde{\Lambda}$ is an antiderivative of $\Lambda$ that is at most linear at infinity and $J_\varepsilon$ is a normalization constant depending on $\varepsilon$. 
Define time-shift operators $\theta_\alpha$ by

$$(\theta_\alpha g)(X_t^\varepsilon) = g(X_{t-\alpha}^\varepsilon)$$

for any measurable function $g$ and any positive number $\alpha$. 
Now, we apply the Itô-formula to $N^\varepsilon_t$ to obtain

\[
dN^\varepsilon_t = d(e^{-rt} Q^\varepsilon(t, X^\varepsilon_t))
\]

\[
= e^{-rt} \left( \frac{\partial}{\partial t} Q^\varepsilon(t, X^\varepsilon_t) + \frac{1}{2} f^2(Y^\varepsilon_t) \eta^2(X^\varepsilon_{t-b})(X^\varepsilon_t)^2 \frac{\partial^2}{\partial x^2} Q^\varepsilon(t, X^\varepsilon_t) \\
+ r \xi(X^\varepsilon_t) X^\varepsilon_{t-a} \frac{\partial}{\partial x} Q^\varepsilon(t, X^\varepsilon_t) - r Q^\varepsilon(t, X^\varepsilon_t) \right) dt
\]

\[
+ e^{-rt} f(X^\varepsilon_t) \eta(X^\varepsilon_{t-b}) X^\varepsilon_t \frac{\partial Q^\varepsilon}{\partial x}(t, X^\varepsilon_t) dW^*_t
\]
Define a function $f_\alpha$ on the OU process $Y_t^\varepsilon$ for any positive number $\alpha$ as following:

$$f_\alpha(Y_t^\varepsilon) = (\theta_\alpha f)(Y_t^\varepsilon) = f(Y_{t-\alpha}^\varepsilon) \quad \text{i.e, } f_\alpha = \theta_\alpha f$$

And, define functions $\xi_\alpha$ & $\eta_\alpha$ on the DSV process $X_t^\varepsilon$ for any positive number $\alpha$ as following:

$$\xi_\alpha(X_t^\varepsilon) = (\theta_\alpha \xi)(X_t^\varepsilon) = \xi(X_{t-\alpha}^\varepsilon) \quad \text{i.e, } \xi_\alpha = \theta_\alpha \xi$$

$$\eta_\alpha(X_t^\varepsilon) = (\theta_\alpha \eta)(X_t^\varepsilon) = \eta(X_{t-\alpha}^\varepsilon) \quad \text{i.e, } \eta_\alpha = \theta_\alpha \eta$$
For convenience, we define an operator \( \mathcal{L}_{DSV}(\bar{\sigma}) \) as follows:

\[
\mathcal{L}_{DSV}(\bar{\sigma}) = \frac{\partial}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \eta^2_b(x) x^2 \frac{\partial^2}{\partial x^2} + r \xi(a(x)x \frac{\partial}{\partial x} - r).
\]

where \( i(x) = x \). Then \( dN_t^\varepsilon \) becomes

\[
dN_t^\varepsilon = e^{-rt} \left( \mathcal{L}_{DSV}(\bar{\sigma}) + \frac{1}{2} (f^2(Y_t^\varepsilon) - \bar{\sigma}^2) \eta^2(X_{t-b}^\varepsilon)(X_t^\varepsilon)^2 \right)
\]

\[
\times \frac{\partial^2 Q^\varepsilon}{\partial x^2}(t, X_t^\varepsilon) dt
\]

\[
+ e^{-rt} f(X_t^\varepsilon) \eta(X_{t-b}^\varepsilon) X_t^\varepsilon \frac{\partial Q^\varepsilon}{\partial x}(t, X_t^\varepsilon) dW_t^*
\]  

(1)
Some definitions

We will find a function $Q^\varepsilon$ satisfying the condition (ii) assumed in Theorem 2. Before doing that, we need some definitions as follows:

**Firstly, $\tilde{P}_0^\varepsilon$**

Define $\tilde{P}_0^\varepsilon$ as:

The solution $\tilde{P}_0^\varepsilon$ of $\mathcal{L}_{DSV}(\overline{\sigma}^\varepsilon)\tilde{P}_0^\varepsilon = 0$

with the terminal condition $\tilde{P}_0^\varepsilon(T, x) = h(x)$

We will call $\mathcal{L}_{DSV}(\overline{\sigma}^\varepsilon)\tilde{P}_0^\varepsilon = 0$ as "Delayed Stochastic Volatility Equation (DSVE)"
Secondly, $V$ and $U$

Define a 2-variable function $V$ and $U$ follows:

$$V(t, x) = \frac{\sqrt{\varepsilon \nu \rho}}{\sqrt{2}} \langle f\phi' \rangle \varepsilon \eta_b^3(x) x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 \tilde{P}_0}{\partial x^2} \right)(t, x)$$

$$U(t, x) = \sqrt{\varepsilon \nu \rho} \sqrt{2} \langle f_b \phi' \rangle \varepsilon \eta_b(x) \eta'_b(x) i_b(x) \eta_{2b}(x) x^2 \frac{\partial^2 \tilde{P}_0}{\partial x^2}(t, x)$$
Thirdly, $\tilde{Q}_1^\varepsilon$

Define $\tilde{Q}_1^\varepsilon$ as below:

\[
\text{the solution } \tilde{Q}_1^\varepsilon \text{ of } \mathcal{L}_{DSV}(\bar{\sigma}_\varepsilon) \tilde{Q}_1^\varepsilon = V + U
\]  

That is, $\mathcal{L}_{DSV}(\bar{\sigma}_\varepsilon) \tilde{Q}_1^\varepsilon(t, X_t^\varepsilon)$

\[
= \frac{\sqrt{\varepsilon \nu \rho}}{\sqrt{2}} \langle f \phi' \rangle_\varepsilon \eta^3(X_{t-b}^\varepsilon) X_t^\varepsilon \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 \tilde{P}_0^\varepsilon}{\partial x^2} \right)(t, X_t^\varepsilon)
\]

\[
+ \sqrt{\varepsilon \nu \rho} \sqrt{2} \langle f_b \phi' \rangle_\varepsilon \eta(X_{t-b}^\varepsilon) \eta'(X_{t-b}^\varepsilon) \eta(X_{t-2b}^\varepsilon)
\]

\[
\times (X_t^\varepsilon)^2 X_{t-b}^\varepsilon \frac{\partial^2 \tilde{P}_0^\varepsilon}{\partial x^2}(t, X_t^\varepsilon)
\]
It’s time to choose $Q^\varepsilon$ satisfying the conditions assumed in Theorem 2. We define $Q^\varepsilon$ as

$$Q^\varepsilon = \tilde{P}_0^\varepsilon + \tilde{Q}_1^\varepsilon$$

(3)

It remains to show the chosen $Q^\varepsilon$ satisfies the desired conditions.

From now, we will confirm it. For that, we need some properties. The following lemmas are helpful for the proof.
Some Lemmas

Define $\phi$ as the solution of

$$L^\varepsilon_Y \phi (y) = f^2 (y) - \langle f^2 \rangle \varepsilon$$

Lemma 1

Let $f$ be a sufficiently smooth function

Then $\int_0^t \left( f^2 (Y^\varepsilon_s) - \sigma^2 \varepsilon \right) ds = O(\sqrt{\varepsilon})$
Lemma 2

Let \( f \) and \( g \) be a sufficiently smooth function. Then

\[
\int_0^t e^{-rs} \left( f(Y_s^\varepsilon)^2 - \sigma_s^2 \right) \eta^2 (X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 \tilde{P}_0}{\partial x^2} (s, X_s^\varepsilon) ds
\]

\[
= \sqrt{\varepsilon} (\tilde{B}_t^\varepsilon + \tilde{M}_t^\varepsilon) + \mathcal{O}(\varepsilon)
\]
where $\overline{B}_t^\varepsilon$ is a systemic bias given by

$$\overline{B}_t^\varepsilon = \sqrt{2\nu \rho} \int_0^t e^{-rs} f(Y_s^\varepsilon) \phi'(Y_s^\varepsilon) \eta^3(X_{s-b}^\varepsilon) X_s^\varepsilon \frac{\partial}{\partial x} (x^2 \frac{\partial^2 \tilde{P}_0^\varepsilon}{\partial x^2}) ds$$

$$+ 2\sqrt{2\nu \rho} \int_0^t e^{-rs} f_b(Y_s^\varepsilon) \phi'(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon) \eta'(X_{s-b}^\varepsilon) \eta(X_{s-2b}^\varepsilon)$$

$$\times (X_s^\varepsilon)^2 X_{s-b}^\varepsilon \frac{\partial^2 \tilde{P}_0^\varepsilon}{\partial x^2} ds$$

(4)

and $\overline{M}_t^\varepsilon$ is a martingale given by

$$\overline{M}_t^\varepsilon = \sqrt{2\nu} \int_0^t e^{-rs} \phi'(Y_s^\varepsilon) \eta^2(X_{s-b}^\varepsilon) (X_s^\varepsilon)^2 \frac{\partial^2 \tilde{P}_0^\varepsilon}{\partial x^2} d\hat{Z}_s$$

(5)
Lemma 2,, continued

**Proof**

The following facts hold. (detailed proofs are omitted)

1. \( (f^2(Y^\varepsilon) - \langle f^2 \rangle^\varepsilon)ds = \varepsilon d\phi(Y^\varepsilon) - \nu \sqrt{2\varepsilon}\phi'(Y^\varepsilon)d\hat{Z}_s^* \)

2. \( \int_0^t e^{-rs} \left( f(Y^\varepsilon)^2 - \sigma^2 \right) g^2(X^\varepsilon_{s-b})(X^\varepsilon_s)^2 \frac{\partial^2 \tilde{P}^\varepsilon_0}{\partial x^2}(s, X^\varepsilon_s)ds \)

\[= \varepsilon \int_0^t e^{-rs} g^2(X^\varepsilon_{s-b})(X^\varepsilon_s)^2 \frac{\partial^2 \tilde{P}^\varepsilon_0}{\partial x^2} d\phi(Y^\varepsilon) \]

\[-\nu \sqrt{2\varepsilon} \int_0^t e^{-rs} g^2(X^\varepsilon_{s-b})\phi'(Y^\varepsilon_s)(X^\varepsilon_s)^2 \frac{\partial^2 \tilde{P}^\varepsilon_0}{\partial x^2}(s, X^\varepsilon_s)d\hat{Z}_s^* \]
Lemma 2, continued

proof, continued

3. $\varepsilon \int_{0}^{t} e^{-st} g^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 Q^\varepsilon}{\partial x^2}(t, X_s^\varepsilon)d\phi(Y_s^\varepsilon) = B_t^\varepsilon + O(\varepsilon)$

Putting these facts together yields Lemma 2.
From the SDDE (1), we obtain $N_t^\varepsilon$ as follows:

$$N_t^\varepsilon = N_0^\varepsilon + \int_0^t e^{-rt} L_{DSV}(\overline{\sigma}_^\varepsilon) Q^\varepsilon ds + \frac{1}{2} \int_0^t e^{-rs} \left( f^2 (Y_s^\varepsilon) - \overline{\sigma}_s^\varepsilon \right) \eta^2 (X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 Q^\varepsilon}{\partial x^2} ds + \int_0^t e^{-rs} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon) X_s^\varepsilon dW_s^*$$

(6)
The decomposition of $N_t^\varepsilon$

By the definition (3) and the definition of $\tilde{P}_0^\varepsilon$,

$$\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)Q^\varepsilon = \mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)\tilde{Q}^\varepsilon$$

Then (6) becomes:

\[
N_t^\varepsilon = N_0^\varepsilon \\
+ \int_0^t e^{-rt} \mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)\tilde{Q}^\varepsilon \, ds \\
+ \frac{1}{2} \int_0^t e^{-rs} (f^2(Y_s^\varepsilon) - \overline{\sigma}_\varepsilon^2) \eta^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 \tilde{Q}^\varepsilon}{\partial x^2} \, ds \\
+ \frac{1}{2} \int_0^t e^{-rs} (f^2(Y_s^\varepsilon) - \overline{\sigma}_\varepsilon^2) \eta^2 (X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 \tilde{P}_0^\varepsilon}{\partial x^2} \, ds \\
+ \int_0^t e^{-rs} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon)X_s^\varepsilon \, dW_s^* 
\]
Also, by Lemma 2, the above $N_t^\varepsilon$ become:

\[
N_t^\varepsilon = N_0^\varepsilon + \int_0^t e^{-rs} \mathcal{L}_{DSV}(\sigma^\varepsilon) \tilde{Q}_1^\varepsilon ds + \sqrt{\varepsilon} \left( B_t^\varepsilon + M_t^\varepsilon \right) + R_1(\varepsilon) \\
+ \int_0^t e^{-rs} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon) X_s^\varepsilon dW_s^*
\]

where $R_1(\varepsilon) = O(\varepsilon)$
By the definitions (2) and (4), we obtain $N_t^\varepsilon$ as follows:

$$
N_t^\varepsilon = N_0^\varepsilon 
+ \frac{\sqrt{\varepsilon \nu \rho}}{\sqrt{2}} \int_0^t e^{-\varepsilon s} (f(Y_s^\varepsilon)\phi'(Y_s^\varepsilon) - \langle f\phi' \rangle^\varepsilon) \eta^3(X_s^\varepsilon) \eta \left( X_s^\varepsilon - b \right)
\times \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 \tilde{P}_t^\varepsilon}{\partial x^2} \right) ds
+ \frac{\sqrt{\varepsilon 2\nu \rho}}{\sqrt{2}} \int_0^t e^{-\varepsilon s} (f_b(Y_s^\varepsilon)\phi'(Y_s^\varepsilon) - \langle f_b\phi' \rangle^\varepsilon) \eta^3(X_s^\varepsilon) \eta \left( X_s^\varepsilon - b \right)
\times \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 \tilde{P}_t^\varepsilon}{\partial x^2} \right) ds
+ \frac{\sqrt{\varepsilon}}{2} \tilde{M}_t + R_2(\varepsilon)
+ \int_0^t e^{-\varepsilon s} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_s^\varepsilon) X_s^\varepsilon dW^*_s
$$

(7)

where $R_2(\varepsilon) = O(\varepsilon)$
Here, as in Lemma 1, the second term and the third term of (7) are included in \( O(\varepsilon) \) can be shown to be of order \( \varepsilon \). So, we have

\[
N_t^\varepsilon = N_0^\varepsilon + \frac{\sqrt{\varepsilon}}{2} M_t^\varepsilon + R_3(\varepsilon)
\]

\[
+ \int_0^t e^{-rs} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon) X_s^\varepsilon dW_s^*
\]

where \( R_3(\varepsilon) = O(\varepsilon) \)
Define $\tilde{M}_t^\varepsilon$ & $R_t^\varepsilon$

\[ \tilde{M}_t^\varepsilon = N_0^\varepsilon + \frac{\sqrt{\varepsilon}}{2} M_t^\varepsilon + \int_0^t e^{-rs} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon) X_s^\varepsilon dW_s^* \]

\[ R_t^\varepsilon = R_3(\varepsilon) \]

Here, $M_t^\varepsilon$ is a martingale and $\int_0^t e^{-rs} \frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon) \eta(X_{s-b}^\varepsilon) X_s^\varepsilon dW_s^*$ is a martingale by Martingale Representation Theorem. Hence, $\tilde{M}_t^\varepsilon$ is also a martingale.
Then, from (8)

\[ e^{-rt} Q^\varepsilon = N_t^\varepsilon = \tilde{M}_t^\varepsilon + R_t^\varepsilon \]

where \( \tilde{M}^\varepsilon \) is a martingale and \( R_t^\varepsilon \) is of order \( \varepsilon \).

So that we can confirm that the \( Q^\varepsilon \) of our choice satisfies the conditions (i) and (ii) in Theorem 2.

Therefore, by Theorem 2,

\[ P_t^\varepsilon = Q^\varepsilon(t, X_t^\varepsilon) + O(\varepsilon) \]
Leading order term, $\tilde{P}$

strike price=100, $\xi(x) = x^{0.2}$, $\eta(x) = x^{0.001}$
Leading order term, $\tilde{P}$

strike price=100, $\xi(x) = x^{\theta_1}$, $\eta(x) = x^{0.1}$

strike price=100, $\xi(x) = x^{0.2}$, $\eta(x) = x^{\theta_2}$

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Correction term, $\tilde{Q}^{\xi}_1$

strike price=100, $\xi(x) = x^{\theta_1}$, $\eta(x) = x^{0.1}$

strike price=100, $\xi(x) = x^{0.2}$, $\eta(x) = x^{\theta_2}$
Comparison of European call option price for DSV model and Black-Scholes model

strike price=100, $\xi(x) = x^{\theta_1}$, $\eta(x) = x^{\theta_2}$

The "blue" line is a case where the delay term is in only drift term, the "green" line is a case where the delay term is in only volatility term and the "black" line is a case where the delay term is in both terms.
Conclusion

- Introduced a new Non-Markovian Stochastic Volatility model.
- The price by DSV model is more flexible to market than BS model.
- Performed asymptotic analysis.
- Still on-going research - Mathematical rigor, Data fitting, and etc.