Asymptotics of implied volatility in local volatility models

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References

[1] Henri Berestycki, Jérôme Busca, and Igor Florent
Asymptotics and calibration of local volatility models

The Volatility Surface: A Practitioner’s Guide.

Asymptotics of implied volatility in local volatility models

Analysis, Geometry, and Modeling in Finance: Advanced Methods in Option Pricing.
Outline

- Implied volatility in terms of local volatility
  - The heat kernel approach
  - The BBF approximation
  - BBF to higher orders

- One expansion, two approaches
  - Laplace asymptotic formula
  - Expansion of time value

- Numerical tests

- Summary and conclusions
Objective

Given a local volatility process

\[ \frac{dS}{S} = \sigma(S, t) \, dW_t, \]

with \( \sigma(S, t) \) depending only on the underlying level \( S \) and the time \( t \), we want to compute implied volatilities \( \sigma_{bs}(K, T) \) such that

\[ C_{bs}(s, t, K, T, \sigma_{bs}(K, T)) = \mathbb{E} \left[ (S_T - K)^+ | S_t = s \right] \]

or in words, we want to efficiently compute implied volatility from local volatility.
Call price

Let \( p(t, s; t', s') \) be the transition probability density. Then

\[
C(s, t, K, T) = \mathbb{E} \left[ (S_T - K)^+ | S_t = s \right] = \int (s' - K)^+ p(t, s; T, s') ds'
\]

As a function of \( t \) and \( s \), \( p \) satisfies the backward Kolmogorov equation:

\[
Lp := p_t + \frac{1}{2} s^2 \sigma^2(s, t) p_{ss} = 0,
\]

Subindices refer to respective partial derivatives.
Two to approximate

\[ C(s, t, K, T) = \int (s' - K)^+ p(t, s; T, s') ds' \]

- Approximate transition density by heat kernel expansion.
- Approximate the integral.
  - Two approaches for approximating the integral lead to one expansion.
- The smaller the time to maturity, the better the approximation, for both approximations.
Heat kernel expansion

Heat kernel expansion for transition density $p(t, s; t', s')$ when $t' - t$ is small:

$$p(t, s; t', s') \sim \frac{e^{-\frac{d^2(s,s',t)}{2(t'-t)}}}{\sqrt{2\pi(t'-t)}}s'\sigma(s', t') \left[ \sum_{k=0}^{n} H_k(t, s, s')(t' - t)^k \right]$$

- $d(s, s', t) = \left| \int_s^{s'} \frac{d\xi}{\xi\sigma(\xi, t)} \right|$ : geodesic distance between $s$ to $s'$
- $H_0(t, s, s') = \sqrt{\frac{s\sigma(s,t)}{s'\sigma(s',t)}} \exp \left[ \int_s^{s'} \frac{dt(\eta,s',t)}{\eta\sigma(\eta,t)} d\eta \right]$
- $H_i(t, s, s') = \frac{H_0(t,s,s')}{d^i(s,s',t)} \int_s^{s'} \frac{d^{i-1}(\eta,s',t)LH_{i-1}}{H_0(\eta,s',t)a(\eta,t)} d\eta$
Heat kernel expansion for Black-Scholes

Heat kernel expansion for Black-Scholes transition density $p_{bs}(t, s; t', s')$ when $t' - t$ is small:

$$p_{bs}(t' - t, s, s') = \frac{e^{-\frac{d_{bs}^2(s, s')}{{2(t' - t)}}}}{\sqrt{2\pi(t' - t)\sigma_{bs}s'}} \sqrt{\frac{s}{s'}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[ \frac{\sigma_{bs}^2(t' - t)}{8} \right]^k$$

- $d_{bs}(s, s') = \left| \int_s^{s'} \frac{d\xi}{\sigma_{bs}\xi} \right| = \frac{1}{\sigma_{bs}} \left| \log \frac{s'}{s} \right|
- $H_{0bs}(t, s, s') = \sqrt{\frac{s}{s'}}$
Main idea

Implied volatility $\sigma_{bs}$ is defined as the unique solution to

$$C(s, t, K, T) = C_{bs}(s, t, K, T, \sigma_{bs})$$

- Substitute the transition density by the heat kernel expansion for both the model price $C$ and the Black-Scholes price $C_{bs}$
- Expand in terms of $T - t$ on both sides of the resulting equation
- Further expand on Black-Scholes side the implied volatility

$$\sigma_{bs}(K, T) \approx \sigma_{bs,0} + \sigma_{bs,1}(T - t) + \sigma_{bs,2}(T - t)^2$$

- Match the corresponding coefficients
Two approaches

- Directly substitute the transition density by heat kernel expansion to call price. Use Laplace asymptotic formula to approximate the resulting integral.

- Rewrite call price as intrinsic value + time value. Further rewrite time value as an integral of transition density over time, i.e., the Carr-Jarrow formula:

  \[ C(s, t, K, T) = (s - K)^+ + \int_t^T K^2 \sigma^2(K, u)p(s, t; K, u)du \]
Laplace asymptotic method

**Laplace asymptotic formula**

Asymptotic expansion of the integral as $\tau \to 0^+$

$$
\int_0^\infty e^{-\frac{\phi(x)}{\tau}} f(x)dx \sim \tau^2 e^{-\frac{\phi(x^*)}{\tau}} \left[ \frac{\phi'(x^*)}{[\phi'(x^*)]^2} + \left( \frac{\phi'(x^*)}{[\phi'(x^*)]^3} \right)' \right] \tau
$$

Assumptions:
- $f$ is identically zero when $0 \leq x \leq x^*$.
- $\phi$ is increasing in $[x^*, \infty)$.
Laplace asymptotic for call price

Let $\tau = T - t$.

$$C(s, t, K, T) = \int_0^\infty (s - K)^+ p(t, s; T, s') ds'$$

$$\sim \frac{1}{\sqrt{2\pi \tau}} \int_0^\infty (s' - K)^+ \frac{e^{-\frac{d^2(s, s', t)}{2\tau}}}{s'\sigma(s', T)} \sum_{k=0}^n H_k(t, s, s') \tau^k ds'$$

$$= \frac{1}{\sqrt{2\pi \tau}} \sum_{k=0}^n \int_K^\infty e^{-\frac{d^2(s, s', t)}{2\tau}} G_k(t, s, T, s') ds' \cdot \tau^k$$

- $G_k(t, s, T, s') = (s' - K) \frac{H_k(t, s, s')}{s'\sigma(s', T)}$
Laplace asymptotic method

Laplace asymptotic for call price

Assume $s < K$.

\[
\frac{1}{\sqrt{2\pi\tau}} \int_{K}^{\infty} e^{-\frac{d^2(s,s',t)}{2\tau}} G_k(t, s, T, s') ds' \\
\sim \frac{\tau^\frac{3}{2}}{\sqrt{2\pi}} e^{-\frac{d^2}{2T}} \left[ \frac{G'_k}{(dd')^2} + \left( \frac{G'_k}{(dd')^3} \right)' \tau \right],
\]

- $d = d(s, K, t)$, $d' = \frac{\partial d}{\partial s'}(s, K, t)$, and $d'' = \frac{\partial^2 d}{\partial (s')^2}(s, K, t)$
- $G'_k = \frac{\partial G_k}{\partial s'}(t, s, T, K) = \frac{H_k(t,s,K)}{K\sigma(K,T)}$
Laplace asymptotic for call price

Laplace asymptotic for model price:

\[ C(s, t, K, T) \sim \frac{\tau^{3/2}}{\sqrt{2\pi}} e^{-\frac{d^2}{2\tau}} \left[ \frac{G_0'}{(dd')^2} + \left\{ \left( \frac{G_0'(K)}{(dd')^3} \right)' + \frac{G_1'(K)}{(dd')^2} \right\} \tau \right]. \]

- \( d = d(s, K, t), \quad d' = \frac{\partial d}{\partial s'}(s, K, t), \) and \( d'' = \frac{\partial^2 d}{\partial (s')^2}(s, K, t) \)
- \( G'_k = \frac{\partial G_k}{\partial s'}(t, s, T, K) = \frac{H_k(t,s,K)}{K\sigma(K,T)} \)

Laplace asymptotic for Black-Scholes: \( k = \log \frac{K}{s} \)

\[ C_{bs}(s, t, K, T, \sigma_{bs}) \sim \frac{Ke^{-k^2/2}}{\sqrt{2\pi}} e^{-\frac{k^2}{2\sigma_{bs}^2\tau}} \frac{\sigma_{bs}^3}{k^2} \frac{T^{3/2}}{\tau^{3/2}} \left[ 1 - \left( \frac{1}{8} + \frac{3}{k^2} \right) \sigma_{bs}^2 \tau \right] \]
Laplace asymptotic method

**Match the coefficients**

Let \( \sigma_{bs} = \sigma_{bs,0} + \sigma_{bs,1}\tau + \sigma_{bs,2}\tau^2 + \cdots \) and set

\[
e^{-\frac{d^2}{2\tau}} \left[ \frac{G_0'}{(dd')^2} + \left\{ \left( \frac{G_0'(K)}{(dd')^3} \right)' + \frac{G_1'(K)}{(dd')^2} \right\} \tau \right]
\]
\[
= e^{-\frac{k^2}{2\sigma_{bs,0}^2\tau}} \frac{K\sigma_{bs,0}^3}{k^2 e^{\frac{k}{2}}} \left[ 1 - \left( \frac{1}{8} + \frac{3}{k^2} \right) \sigma_{bs,0}^2 \tau \right]
\]

- **Exponential term:** \( d^2 = \frac{k^2}{\sigma_{bs,0}^2} \implies \sigma_{bs,0} = \frac{k}{d} = \frac{\log K - \log s}{d(s,K,t)} \)

- **Zeroth order term:**

\[
\frac{G_0'}{(dd')^2} = e^{\frac{k^2\sigma_{bs,0}^3}{\sigma_{bs,0}^3} \frac{K\sigma_{bs,0}^3}{k^2 e^{\frac{k}{2}}}} \implies \sigma_{bs,1} = \frac{k}{d^3} \log \left[ \frac{dG_0' e^{-\frac{k}{2}}}{Kk(d')^2} \right]
\]
Recall

\[ C(s, t, K, T) = (s - K)^+ + \int_t^T K^2 \sigma^2(K, u)p(s, t; K, u)du \]

\[ \sim (s - K)^+ + \sum_{k=0}^{n} \int_t^T \frac{e^{-\frac{d^2(s,K,t)}{2(u-t)}}}{\sqrt{2\pi(u-t)}} K\sigma(K, u)(u - t)^k du \cdot H_k(t, s, K) \]

Moreover, denote \( d = d(s, K, t) \),

\[ \int_t^T e^{-\frac{d^2}{2(u-t)}} \sigma(K, u)(u - t)^{k-\frac{1}{2}} du \]

\[ \sim \int_t^T e^{-\frac{d^2}{2(u-t)}} [\sigma(K, t) + \sigma_t(K, t)(u - t)](u - t)^{k-\frac{1}{2}} du \]
Expansion for call price

Let $\Phi_k(d, \tau) = \int_0^t u^{k-\frac{1}{2}} e^{-\frac{d^2}{2u}} du$.

$$C(s, t, K, T) - (s - K)^+ \sim \frac{1}{2\sqrt{2\pi}} \left\{ K\sigma(K, t)\Phi_0(d, \tau)H_0(t, s, K) 
+ K[\sigma_t(K, t)H_0(t, s, K) + \sigma(K, t)H_1(t, s, K)]\Phi_1(d, \tau) \right\}$$

Moreover, on Black-Scholes side,

$$C_{bs}(s, t, K, T) - (s - K)^+ \sim \frac{\sqrt{sK}}{\sqrt{2\pi}} \left[ \sigma_{bs} \Phi_0(d_{bs}, \tau) - \frac{\sigma_{bs}^3}{8} \Phi_1(d_{bs}, \tau) \right]$$
Auxiliary expansion and matching

Expanding the $\Phi_i$’s:

- $\Phi_0(d, \tau) \sim 2\tau^{\frac{3}{2}} \left[ \frac{1}{d^2} - 3 \frac{\tau}{d^4} \right] e^{-\frac{d^2}{2\tau}}$

- $\Phi_1(d, \tau) = \frac{2}{3} \tau^{\frac{3}{2}} e^{-\frac{d^2}{2\tau}} - \frac{d^2}{3} \Phi_0(d, \tau) \sim \frac{2\tau^{\frac{5}{2}}}{d^2} e^{-\frac{d^2}{2\tau}}$

Matching

$$e^{-\frac{d^2(s, K, t)}{2\tau}} \left\{ \frac{K \sigma H_0}{d^2} + \left[ \frac{K \sigma_t H_0 + K \sigma H_1}{d^2} - \frac{3K \sigma H_0}{d^4} \right] \tau \right\}$$

$$= e^{-\frac{d_{bs}^2(s, K, t)}{2\tau}} \sqrt{sK} \left[ \sigma_{bs} \Phi_0(d_{bs}, \tau) - \frac{\sigma_{bs}^3}{8} \Phi_1(d_{bs}, \tau) \right]$$
Asymptotic expansion once again

\[ \sigma_{bs} = \sigma_{bs,0} + \sigma_{bs,1}(T - t) + \sigma_{bs,2}(T - t)^2 + \mathcal{O}(T - t)^3. \]

\[ d(s, K, t) = \int_s^K \frac{d\xi}{\xi \sigma(\xi, t)}, \]

\[ H_0(s, K, t) = \sqrt{\frac{s \sigma(s, t)}{K \sigma(K, t)}} \exp \left[ \int_s^K \frac{d_t(\eta, K, t)}{\eta \sigma(\eta, t)} d\eta \right]. \]

\begin{itemize}
  \item \( \sigma_{bs,0} = \frac{|\log K - \log s|}{d(s, K, t)}. \) (BBF)
  \item \( \sigma_{bs,1} = \frac{k}{d^3} \log \left[ \frac{dH_0 \sqrt{K \sigma(K, t)}}{k \sqrt{s}} \right], \) where \( k = \log K - \log s. \)
  \item \( \sigma_{bs,2}? \) Too complicated to reproduce here.
\end{itemize}
Henry-Labordère’s approximation

Henry-Labordère also presents a heat kernel expansion based approximation to implied volatility in equation (5.40) on page 140 of his book [4]:

\[
\sigma_{BS}(K, T) \approx \sigma_0(K) \left\{ 1 + \frac{T}{3} \left[ \frac{1}{8} \sigma_0(K)^2 + Q(f_{av}) + \frac{3}{4} G(f_{av}) \right] \right\}
\]

(1)

with

\[
Q(f) = \frac{C(f)^2}{4} \left[ \frac{C''(f)}{C(f)} - \frac{1}{2} \left( \frac{C'(f)}{C(f)} \right)^2 \right]
\]

and

\[
G(f) = 2 \partial_t \log C(f) = 2 \frac{\partial_t \sigma(f, t)}{\sigma(f, t)}
\]

where \( C(f) = f \sigma(f, t) \) in our notation, \( f_{av} = (S_0 + K)/2 \) and the term \( \sigma_0(K) \) is the BBF approximation from [1].
How well do these approximations work?

We consider the following explicit local volatility models:

- The square-root CEV model:
  \[ dS_t = e^{-\lambda t} \sigma \sqrt{S_t} \, dW_t \]

- The quadratic model:
  \[ dS_t = e^{-\lambda t} \sigma \left\{ 1 + \psi (S_t - 1) + \frac{\gamma}{2} (S_t - 1)^2 \right\} \, dW_t \]

Parameters are: \( \sigma = 0.2, \psi = -0.5 \) and \( \gamma = 0.1 \). In each case \( S_0 = 1 \) and \( T = 1 \).

- \( \lambda = 0 \) gives a time-homogeneous local volatility surface and \( \lambda = 1 \) a time-inhomogeneous one.

- We compare implied volatilities from the approximations and the closed-form solution.
Time-homogeneous Square Root CEV

Note that all errors are tiny!
Time-homogeneous Quadratic Model

Approximation error

H-L
σ_1
σ_2

Strike
Time-inhomogeneous Square Root CEV

![Graph 1](image1)

![Graph 2](image2)
Time-inhomogeneous Quadratic Model
Summary

- Small-time expansions are useful for generating closed-form expressions for implied volatility from simple models.
- Direct substitute approach is easier for generalization to higher dimensions, e.g., stochastic volatility models.

\[ \sigma_{bs} \sim \frac{\log K - \log s}{d_M(s, v)}, \]

where \( d_M(s, v) \) is the “distance to the money”, i.e., shortest geodesic distance from the spot \((s, v)\) to the line \( \{s = K\} \) in the price-volatility plane.

- Application: Short time implied vol in delta is flat!
  (Joint work with Carr and Lee).
Time value approach is easier for getting higher order terms.

Refinement of $\sigma_{bs,0}$ (work in progress with Gatheral):

$$
\sigma_{bs} \sim \left[ \frac{\sqrt{T-t}}{|\log K - \log s|} \sqrt{\int_t^T \left| \frac{s' (\tau)}{a(s(\tau), \tau)} \right|^2 d\tau} \right]^{-1}
$$

where the integral is along the “most likely path” $s(\tau)$.

If we take the “likely path” as $s(\tau) = \varphi_t^{-1} \left( \frac{\tau}{T} \varphi_t(K) \right)$, where $\varphi_t(x) = \int_s^x \frac{d\xi}{a(\xi, t)}$, then BBF is recovered.
THANK YOU FOR YOUR PATIENCE.