When are path-dependent payoffs suboptimal?

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When are path-dependent payoffs suboptimal?

Introduction
   Answers in special settings
   Questions

Attractive payoffs
   When path-independent payoffs are preferred
   When increasing payoffs are preferred
   In a nutshell

Examples
   Geometric Brownian motion
   Independent increments

Summary
Path-dependent payoffs are known to be suboptimal e.g. in


- and more generally in all

- Exponential Lévy models with Esscher transform, where favorable path-independent payoffs for risk-averse investors are constructed in Vanduffel, Chernih, Maj and Schoutens 2009 A note on the suboptimality of path-dependent payoffs in general markets. Applied Mathematical Finance 16(4).
When do risk-averse investors prefer path-independent payoffs?

Why?

Exponential Lévy model?

Esscher transform?

When are path-dependent payoffs suboptimal?
Setting the stage

- Discounted payoff $X$ at investment horizon $T$
- Investor is strongly risk-averse
  - concave stochastic order, i.e.
  - $Y$ preferable to $X$ if $E[U(Y)] \geq E[U(X)]$ for all concave functions $U$
- Pricing kernel $Z$, i.e.
  - payoff $X$ at time $T$
  - price $E[Z_T X]$ today
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- No specific assumptions on stochastic model and pricing kernel so far.
- No utility function specified, just strong risk aversion.
Risk-averse investors like conditioning

If $X$ is $\mathcal{F}_T$-measurable and $\mathcal{G}$ is a sub-sigma-algebra

$$\mathbb{E}[U(\mathbb{E}[X|\mathcal{G}])] \geq \mathbb{E}[U(X)]$$

for all concave functions $U$ (due to Jensen’s inequality).

$\implies$ Our investor prefers $\mathbb{E}[X|\mathcal{G}]$ over $X$. 

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But can she actually afford $\mathbb{E}[X|\mathcal{G}]$?

That means, is $\mathbb{E}\left[Z_T \mathbb{E}[X|\mathcal{G}] \right] \leq \mathbb{E}[Z_T X]$ for some $\mathcal{G}$?
Attractive payoffs

When path-independent payoffs are preferred

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Yes, she can afford $\mathbb{E}[X|Z_T]$!
Risk-averse investors like conditioning
Functions of kernel preferred

- Risk averse investors prefer $\mathbb{E}[X|Z_T]$ over $X$.
- $\mathbb{E}[X|Z_T]$ and $X$ have the same price.
- Moreover, if we define $\mathbb{Q}$ via $d\mathbb{Q} = Z_T d\mathbb{P}$, $\mathbb{E}_\mathbb{Q}[X|Z_T] = \mathbb{E}[X|Z_T]$. 

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Therefore

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In particular

- If $(S_t)$ is the discounted underlying and $Z_T = g(S_T)$ is an injective function of $S_T$,
- the path-independent payoff $\mathbb{E}[X|S_T]$ is preferable.
Cheaper payoff with same distribution

Basic example

- Suppose that there are only two states $\omega_1$ and $\omega_2$, each occurring with probability one half.
- Let $S_T(\omega_1) = 1$ and $S_T(\omega_2) = 2$.
- Assume $g(1) > g(2)$.

Price of payoff $h(S_T)$ is $\frac{1}{2} g(1) h(1) + \frac{1}{2} g(2) h(2)$
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The following payoffs have the same distribution

- $h(1) = 2$ | $h(1) = 1$
- $h(2) = 1$ | $h(2) = 2$

but the increasing payoff is cheaper

$$g(1) + \frac{1}{2}g(2) > \frac{1}{2}g(1) + g(2)$$

When are path-dependent payoffs suboptimal?
Cheaper payoff with same distribution
Increasing payoffs preferable in case of decreasing kernel

If \( Z_T = g(S_T) \) is a decreasing function of \( S_T \) and
    \( F \) is the distribution function of \( S_T \)
    \( H^- \) is the inverse distribution function of \( h(S_T) \)
    \( \hat{h}(x) := H^-(F(x)) \),

\[ \implies \hat{h} \) is increasing and
\[ \hat{h}(S_T) = H^-(F(S_T)) \) is distributed as \( h(S_T) \)
but \( \hat{h}(S_T) \) is cheaper.

I.e. \( \hat{h}(S_T) \) is preferable.
Attractive payoffs

- Risk-averse investors prefer payoffs of type \( \tilde{h}(Z_T) \).
  Distribution \( \sim \tilde{h}(Z_T) = \mathbb{E}[X|Z_T] \).

- If \( Z_T = g(S_T) \) is an injective function of \( S_T \), path-independent payoffs \( h(S_T) \) are preferred.
  Distribution \( \sim h(S_T) = \mathbb{E}[X|S_T] \).

- If \( g \) is decreasing, \( \hat{h}(S_T) \) with increasing \( \hat{h} \) are preferred.
  Distribution \( \sim \hat{h}(S_T) := H^{-1}(F(S_T)) \).

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Attractive payoffs

- Risk-averse investors prefer payoffs of type $\tilde{h}(Z_T)$. Distribution $\rightsquigarrow \tilde{h}(Z_T) = \mathbb{E}[X|Z_T]$.

- If $Z_T = g(S_T)$ is an injective function of $S_T$, path-independent payoffs $h(S_T)$ are preferred. Distribution $\rightsquigarrow h(S_T) = \mathbb{E}[X|S_T]$.

- If $g$ is decreasing, $\hat{h}(S_T)$ with increasing $\hat{h}$ are preferred. Distribution $\rightsquigarrow \hat{h}(S_T) := H^{-1}(F(S_T))$.

Esscher transform: $Z_T = g(S_T) = \frac{S_T^\gamma}{\mathbb{E}[S_T^\gamma]}$
Attractive payoffs

- Risk-averse investors prefer payoffs of type \( \hat{h}(Z_T) \). Distribution \( \sim \hat{h}(Z_T) = \mathbb{E}[X|Z_T] \).

- If \( Z_T = g(S_T) \) is an injective function of \( S_T \), path-independent payoffs \( h(S_T) \) are preferred. Distribution \( \sim h(S_T) = \mathbb{E}[X|S_T] \).

- If \( g \) is decreasing, \( \hat{h}(S_T) \) with increasing \( \hat{h} \) are preferred. Distribution \( \sim \hat{h}(S_T) := H^{-}(F(S_T)) \).

If (increasing and convex) utility function \( U \) is known, \( \hat{h}(Z_T) \) is optimal if \( Z_T \propto U'(\hat{h}(Z_T)) \).

If \( Z_T = g(S_T) \) and \( U' \propto g \circ h^{-} \), \( h(S_T) \) is optimal.

When are path-dependent payoffs suboptimal?
Conditional expectation
for Geometric Brownian motion

\[ S_t = e^{\sigma B_t + \mu t} \]

with a \( \mathbb{P} \)-standard Brownian motion \( (B_t) \), and let \( g \) be a deterministic function. Then,

\[
\mathbb{E} \left[ e^{\int_0^T g(u) \, d(\log S_u)} \middle| S_T \right] = S_T \exp \left( \frac{1}{T} \int_0^T g(u)^2 \, du - \left( \frac{1}{T} \int_0^T g(u) \, du \right)^2 \right).
\]
Early payment and geometric average for Geometric Brownian motion

\[ S_t = e^{\sigma B_t + \mu t} \]

**Early payment**

\[ \mathbb{E} \left[ S_T^\lambda \left| S_T \right. \right] = S_T^{\lambda \frac{u}{T}} e^{\lambda^2 \frac{\sigma^2 u}{2} \left( 1 - \frac{u}{T} \right)} \]

**Continuous geometric average**

\[ \mathbb{E} \left[ e^{\frac{1}{T} \int_0^T (\log S_u) \, du} \left| X_T \right. \right] = \mathbb{E} \left[ e^{\int_0^T \left( 1 - \frac{u}{T} \right) d(\log S_u)} \left| X_T \right. \right] = \sqrt{S_T} e^{\frac{\sigma^2 T}{24}} \]

When are path-dependent payoffs suboptimal?
Supremum for Geometric Brownian motion

\[ S_t = e^{\sigma B_t + \mu t} \]

The supremum of \((S_u)\) on \([0, T]\) conditional on the terminal value \(S_T\) is

\[
\mathbb{E} \left[ \sup_{u \leq T} S_u \left| S_T \right. \right] = (S_T \vee 1) \left\{ 1 + \frac{\sigma \sqrt{T} \Phi \left( -\frac{\log S_T}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} \right)}{2 \varphi \left( -\frac{\log S_T}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} \right)} \right\}.
\]

Here, \(\varphi\) is the density function and \(\Phi\) is the cumulative distribution function of a standard normal distribution.
Conditional expectation for independent increments

Suppose that $L$ is a process such that the increments $L_s - L_0$ and $L_t - L_s$ are independent.
Moreover, suppose that the distribution of these increments admits densities $f_{0,s}$ and $f_{s,t}$. Then,

$$
E[c(L_s)|L_T] = h_c(L_T), \quad \text{where} \quad h_c(x) = \frac{\int_{-\infty}^{\infty} c(y) f_{0,s}(y) f_{s,T}(x - y) \, dy}{\int_{-\infty}^{\infty} f_{0,s}(y) f_{s,T}(x - y) \, dy}
$$

$$
E[c(L_s) f_{s,T}(x - L_s)] = \frac{E[c(L_s) f_{s,T}(x - L_s)]}{E_P[f_{s,T}(x - L_s)]}.
$$

When are path-dependent payoffs suboptimal?
Summary

- If the pricing kernel is path-independent, risk-averse investors prefer path-independent payoffs.
  - E.g. exponential Lévy with Esscher transform.
  - However, a path-independent net position can consist of several path-dependent payoffs.

- On the other hand, if the pricing kernel is path-dependent, path-dependent payoffs are attractive.
  - E.g. exponential Lévy with minimal entropy martingale measure, q-optimal martingale measure etc.
  - However, complex path-dependent products may not be available at a competitive price.