Optimal Stock Selling Based on the Global Maximum

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(joint work with Dr. M. Dai and Z. Yang)
If I were an Innocent Investor...

- I just bought a stock and must sell it in one year
- Need to decide when to sell?
- Obviously, sell it at the maximum price of the whole year. However, this is an impossible mission.
- So, what about selling at the price "closest" to the maximum?
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- So, what about selling at the price "closest" to the maximum?
- This talk is using square error to measure "closeness" and studying the optimal selling strategy under this criterion.
The Model

- A Black-Scholes market with one stock and one saving account
- The \textit{discounted} stock price follows, on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu \in (-\infty, \infty)$ and $\sigma > 0$ are constants
- Let $M_s = \max_{0 \leq t \leq s} S_t$, $0 \leq s \leq T$ be the running maximum of stock price
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- Let $M_s = \max_{0 \leq t \leq s} S_t$, $0 \leq s \leq T$ be the running maximum of stock price
- Consider the following optimal stopping problem
  \[ \inf_{0 \leq \nu \leq T} \mathbb{E}[(S_\nu - M_T)^2], \]
  where $\mathbb{E}$ stands for the expectation, $\nu$ is an $\mathcal{F}_t$-stopping time.
Graversen, Peskir and Shiryaev (2000), Theory Prob Appl, studied

\[ \inf_{0 \leq \nu \leq T} \mathbb{E}[(S_\nu^0 - M_T^0)^2], \]

where \( S_t^0 = W_t, M_T^0 = \max_{0 \leq t \leq T} W_t \) and obtained explicit optimal solution

\[ \nu^* = \inf \{ t : M_t^0 - S_t^0 \geq z^* \sqrt{T - t} \}, z^* = 1.12 \ldots \]
Related (Probabilistic) Literature

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$$\nu^* = \inf \{ t : M_t^0 - S_t^0 \geq z^* \sqrt{T - t} \}, z^* = 1.12 \ldots$$

- du Toit and Peskir (2007), Ann Prob, considered

$$\inf_{0 \leq \nu \leq T} \mathbb{E}[(S_\nu^\mu - M_T^\mu)^2],$$

where $\mu \neq 0$. 

Shiryaev, Xu and Zhou (2008), Quant Fin, studied the relative error between the selling price and global maximum,

\[
\inf_{0 \leq \nu \leq T} \mathbb{E} \left[ \frac{S_{\nu}}{M_T} \right]
\]

"Bang-bang" strategy:
- Sell at time \( T \) : \( \mu > \frac{\sigma^2}{2} \)
- Sell at time 0 : \( \mu \leq \frac{\sigma^2}{2} \)
PDE Formulation

- The problem is
  \[ \inf_{0 \leq \nu \leq T} \mathbb{E}[(S_{\nu} - M_T)^2] \]

- Not a standard optimal stopping problem, since \( M_T \) is not \( \mathcal{F}_t \)-adapted

- One more step:
  \[
  \inf_{0 \leq \nu \leq T} \mathbb{E}[(S_{\nu} - M_T)^2] = \inf_{0 \leq \nu \leq T} \mathbb{E}\left\{ \mathbb{E}[(S_{\nu} - M_T)^2 \mid \mathcal{F}_\nu] \right\}
  = \inf_{0 \leq \nu \leq T} \mathbb{E}\left\{ \phi(\nu, S_{\nu}, M_{\nu}) \right\},
  \]

where \( \phi(t, S_t, M_t) = \mathbb{E}[(S_t - M_T)^2 \mid \mathcal{F}_t] \)
PDE Formulation (Con’t)

- Denote the value function

\[ \psi(t, S_t, M_t) = \inf_{t \leq \nu \leq T} \mathbb{E}\{ \phi(\nu, S_\nu, M_\nu) \mid F_t \} \]

- Dynamic programming equation (Variational Inequalities)

\[
\begin{align*}
\max \{- \partial_t \psi - \mathcal{L}^0 \psi, \psi - \phi\} &= 0, \quad (t, S, M) \in D, \\
\partial_M \psi(t, M, M) &= 0, \quad \psi(T, S, M) = (S - M)^2,
\end{align*}
\]

where \( \mathcal{L}^0 = \frac{\sigma^2}{2} \partial_{SS} + \mu \partial_S \) and
\[ D = \{(t, S, M) : 0 < S < M, \ 0 \leq t < T\} . \]
The Obstacle Function \( \phi(t, S, M) \)

- Recall

\[
\phi(t, S_t, M_t) = \mathbb{E}[(S_t - M_T)^2 | \mathcal{F}_t]
\]
\[
= S_t^2 - 2S_t \mathbb{E}[M_T | \mathcal{F}_t] + \mathbb{E}[M_T^2 | \mathcal{F}_t]
\]
\[
=: S_t^2 - 2S_t \phi_1(t, S_t, M_t) + \phi_2(t, S_t, M_t),
\]
where \( \phi_i(t, S_t, M_t) = \mathbb{E}[M^i_T | \mathcal{F}_t] \).

- Then, \( \phi_i(t, S, M) \) satisfies

\[
\begin{cases}
- \partial_t \phi_i - \mathcal{L}^0 \phi_i = 0, & (t, S, M) \in D, \\
\partial_M \phi_i(t, M, M) = 0, & \phi_i(T, S, M) = M^i.
\end{cases}
\]
Denote $\tau = T - t$, $x = \ln \frac{M}{S}$, $u_i(\tau, x) = \frac{\phi_i(t, S, M)}{S^i}$, $u(\tau, x) = \frac{\phi(t, S, M)}{S^2}$.

Then, $u_1$ and $u_2$ satisfy
\[
\begin{align*}
\partial_\tau u_1 - \mathcal{L}_x^1 u_1 &= 0 \quad \text{in } \Omega, \\
\partial_x u_1(\tau, 0) &= 0, \quad u_1(0, x) = e^x,
\end{align*}
\]
\[
\begin{align*}
\partial_\tau u_2 - \mathcal{L}_x^2 u_2 &= 0 \quad \text{in } \Omega, \\
\partial_x u_2(\tau, 0) &= 0, \quad u_2(0, x) = e^{2x},
\end{align*}
\]
where $\mathcal{L}_x^1 = \frac{\sigma^2}{2} \partial_{xx} - \left( \mu + \frac{\sigma^2}{2} \right) \partial_x + \mu$,
$\mathcal{L}_x^2 = \frac{\sigma^2}{2} \partial_{xx} - \left( \mu + \frac{3\sigma^2}{2} \right) \partial_x + (2\mu + \sigma^2)$,
$\Omega = (0, T] \times (0, \infty)$. 
Denote $v(\tau, x) = \frac{\psi(t,S,M) - \phi(t,S,M)}{S^2}$

Then, $v$ satisfies

$$\max \left\{ \partial_\tau v - \mathcal{L}_x^2 v - H, v \right\} = 0 \quad \text{in } \Omega,$$

$$\partial_x v(\tau, 0) = 0, \quad v(0, x) = 0,$$

where $H = \mathcal{L}_x^2 u - \partial_\tau u = 2\mu + \sigma^2 + 2\left(\sigma^2 \partial_x u_1 - (\mu + \sigma^2) u_1\right)$,

$$\mathcal{L}_x^2 = \frac{\sigma^2}{2} \partial_{xx} - \left(\mu + \frac{3\sigma^2}{2}\right) \partial_x + (2\mu + \sigma^2).$$

Define the selling region (the stopping region) as follows:

$$SR = \{ (\tau, x) \in [0, \infty) \times (0, T) : v(\tau, x) = 0 \}.$$
The Optimal Selling Strategy: Good Stock ($\mu > 0$)

Figure: Two optimal selling boundaries. Parameter values used:
$\mu = 0.045$, $\sigma = 0.3$, $T = 1$. 
The Optimal Selling Strategy: Bad Stock \((-\sigma^2 \leq \mu \leq 0)\)

**Figure:** The monotonically increasing optimal selling boundary. Parameter values used: \(\mu = -0.010, \sigma = 0.3, T = 1\).
The Optimal Selling Strategy: Very Bad Stock ($\mu < -\sigma^2$)

Figure: The nonmonotone optimal selling boundary. Parameter values used: $\mu = -0.032$, $\sigma = 0.4$, $T = 3$. 
The Proof

Recall

\[
\max \{ \partial_\tau v - L^2_x v - H, v \} = 0 \quad \text{in } \Omega,
\]

\[
\partial_x v(\tau, 0) = 0, \quad v(0, x) = 0,
\]

So,

\[
SR = \{ (\tau, x) : v = 0 \}
\subseteq \{ (\tau, x) : \partial_\tau 0 - L^2_x 0 - H \leq 0 \}
= \{ (\tau, x) : H \geq 0 \}
The Set $\{(\tau, x) : H \geq 0\}$

**Lemma:** Recall $H(\tau, x) = 2\mu + \sigma^2 + 2\left(\sigma^2 \partial_x u_1 - (\mu + \sigma^2)u_1\right)$.

- If $\mu \leq 0$, $\partial_x H > 0$;
- If $\mu \geq -\sigma^2$, $\partial_\tau H < 0$;
- If $\mu > 0$, $\partial_x H(\tau, x) = 0$ has at most one solution for any given $\tau > 0$;
The Main Results: $\mu \leq 0$

With the help of previous lemma, we have

- $\partial_x v \geq 0$ if $\mu \leq 0$;
- $\partial_\tau v \leq 0$ if $\mu \geq -\sigma^2$;
- These are due to

$$\partial_\tau v - L^2_x v = H, \text{ in } \{(\tau, x) : v < 0\}.$$
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Define $x^*_s(\tau) = \inf\{x \in (0, +\infty) : v(\tau, x) = 0, \forall \tau \in (0, T]\}$. 
The Main Results: $\mu \leq 0$

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- These are due to $\partial_\tau v - \mathcal{L}^2_x v = H$, in $\{(\tau, x) : v < 0\}$.

- Define $x_s^*(\tau) = \inf\{x \in (0, +\infty) : v(\tau, x) = 0, \forall \tau \in (0, T]\}$.
- Thanks to $\partial_x v \geq 0$, we can show

\[
SR = \{(\tau, x) : v(\tau, x) = 0\} \\
= \{(\tau, x) : x \geq x_s^*(\tau), 0 < \tau \leq T\}.
\]
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With the help of previous lemma, we have

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- $\partial_\tau v \leq 0$ if $\mu \geq -\sigma^2$;
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- Thanks to $\partial_x v \geq 0$, we can show
  
  $$SR = \{(\tau, x) : v(\tau, x) = 0\}$$
  $$= \{(\tau, x) : x \geq x^*_s(\tau), 0 < \tau \leq T\}.$$ 

- $\partial_\tau v \leq 0$ gives the monotonicity of the free boundary.
The Main Results: $\mu > 0$

- With $\mu > 0$, we have $\partial_\tau v \leq 0$, which implies that $(\tau_2, x) \in SR$, if $(\tau_1, x) \in SR$ and $\tau_2 < \tau_1$. 

![Diagram](image-url)
The Main Results: $\mu > 0$

- With $\mu > 0$, we have $\partial_\tau v \leq 0$, which implies that $(\tau_2, x) \in SR$, if $(\tau_1, x) \in SR$ and $\tau_2 < \tau_1$. 

![Diagram showing the relationship between $\tau$, $x$, and $v$ with regions H>0 and H<0.](image)
The Main Results: \( \mu > 0 \)

- With \( \mu > 0 \), we have \( \partial_\tau v \leq 0 \), which implies that \((\tau_2, x) \in SR\), if \((\tau_1, x) \in SR\) and \(\tau_2 < \tau_1\).
The Main Results: $\mu > 0$

- With $\mu > 0$, we have $\partial_\tau v \leq 0$, which implies that $(\tau_2, x) \in SR$, if $(\tau_1, x) \in SR$ and $\tau_2 < \tau_1$. 

![Diagram showing regions H>0 and H<0 with points (\tau_1, x) and (\tau_2, x) in SR]
The Main Results: $\mu > 0$

- The sell region $SR$ is connected;
- We can define
  
  $x_{1s}^*(\tau) = \inf\{x \in [0, +\infty) : v(\tau, x) = 0\}$
  
  $x_{2s}^*(\tau) = \sup\{x \in [0, +\infty) : v(\tau, x) = 0\}$

- It is easy to show
  
  $SR = \{(\tau, x) : x_{1s}^*(\tau) \leq x \leq x_{2s}^*(\tau), 0 < \tau \leq \tau^*\}.$

- The monotonicity of $x_{is}^*(\tau)$ follows by $\partial_\tau v \leq 0.$
Smoothness of the Free Boundary

For $\mu \geq -\sigma^2$, we have $\partial_\tau v \leq 0$. So, one can easily establish the smoothness of $x^*_s(\tau)$ following Friedman (1975).

- First, show $x^*_s(\tau) \in C^{3/4}((0, T])$
- Then, show $x^*_s(\tau) \in C^1((0, T])$
- By a bootstrap argument, show $x^*_s(\tau) \in C^\infty((0, T])$
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- For $\mu < -\sigma^2$, we change of variables. Let $y = x - \mu/\sigma^2 \tau$, and $V(\tau, y) = v(\tau, x)$.
  - Show $\partial_\tau V(\tau, y) \leq 0$ and $\partial_y V(\tau, y) \geq 0$
  - Apply Friedman (1975) to show smoothness of the corresponding $y^*_s(\tau)$, which gives the desired result
Conclusion

- We examine the optimal decision to sell a stock with the criteria of minimizing the square error between the selling price and the global maximum.
- For good stock, i.e. $\mu > 0$, the optimal selling boundary has two branches and only exists when time to maturity is not long enough.
- For bad stock, i.e. $\mu \leq 0$, the optimal selling boundary only has one branch and always exists.
- The smoothness of the free boundary is also established.
Thank you!