Large Traders and Illiquid Options: Hedging vs. Manipulation

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In practice, we observe particularly large trading activities when derivatives mature ("witches’ sabbaths").

Another example for a price impact: the battle for control of Volkswagen

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Research questions

Given this empirical evidence, what are the **optimal manipulation strategies** of large traders with price impact that hold/issued illiquid derivatives?

What is the large trader’s **indifference price** (reservation price) of an illiquid derivative?

Extensive literature on price impact models:


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Model considered in Kraft and K. (2010)

- Investment opportunities of large trader
  1. money market account with zero interest
  2. risky small cap stock $S$, whose drift rate is affected by the €-amount $(\theta_t)_{t\in[0,T]}$ the large trader holds in stocks.

- stock dynamics: $dS_t = S_t [(\mu_0 + \mu_1 \theta_t) \, dt + \sigma \, dW_t]$
  typically: $\mu_1 < 0$, “squeezing” ($\mu_1 > 0$, “herding”)

- Justified as equilibrium stock price process by DeMarzo and Urošević (2006)

- This leads to the gain process $X$ given by $X_0 = 0$ and
  $$dX_t = \frac{\theta_t}{S_t} \, dS_t = \theta_t (\mu_0 + \mu_1 \theta_t) \, dt + \theta_t \sigma \, dW_t$$

- Moreover, large trader issues an illiquid derivative on the stock with time $T$ payoff $h(S_T)$ (“over the counter”)

- total wealth at time $T = p^h - h(S_T) + X_T$

- To switch from seller’s to buyer’s viewpoint replace $h$ by $-h$. 

Holger Kraft, Christoph Kühn
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- **Immediate observation**: despite of the price impact $\mu_1 \neq 0$ the large trader can perfectly replicate the claim $h(S_T)$ at the same costs as in the corresponding standard Black-Scholes model with $\mu_1 = 0$.

- *One* explanation: distribution of price process under “martingale measure” does not depend on $(\theta_t)_{t \in [0,T]}$.

  Replication costs $= \text{expected payoff under martingale measure}$

  $\leadsto$ we have the reference Black-Scholes hedge $\theta^{BS}$ and price $p^{BS}$

But due to the price impact there appears a **trade-off**:

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Utility-based hedging and indifference pricing

- Exponential utility: \( u(Y) = E[-\exp(-\alpha Y)], \ \alpha > 0 \) risk aversion

- \( p^h \) is the **seller’s indifference price** for the derivative payoff \( h(S_T) \) iff

\[
\sup_{\theta} E[-\exp(-\alpha(p^h - h(S_T(\theta)) + X_T(\theta)))] = \sup_{\theta} E[-\exp(-\alpha(X_T(\theta)))]
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Utility with derivative deal \( \stackrel{!}{=} \) Utility without derivative deal

- **New**: \( h(S_T(\theta)) \) depends on \( \theta \).
  \( X_T(\theta) \) is no longer linear in the strategy \( \theta \)
  \( \implies \) in general \( p^h \neq p^{BS} \)

**Hedging manipulation strategy** := \( \widehat{\theta}(\text{with claim}) - \widehat{\theta}(\text{without claim}) \)
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**Hedging manipulation strategy** := \( \hat{\theta} \) (with claim) − \( \hat{\theta} \) (without claim)
Assume that $\mu_1 < \frac{1}{2} \alpha \sigma^2$. Large trader’s value function:

$$G(t, x, s) = \sup_{\theta} E \left[ -\exp(-\alpha(-h(S_T(\theta)) + X_T(\theta))) \right]$$

has to satisfy Hamilton-Jacobi-Bellman equation

$$\max_{\vartheta \in \mathbb{R}} \left\{ G_t + \vartheta (\mu_0 + \mu_1 \vartheta) G_x + (\mu_0 + \mu_1 \vartheta) s G_s + \frac{1}{2} \sigma^2 \vartheta^2 G_{xx} + \frac{1}{2} \sigma^2 s^2 G_{ss} + \sigma^2 \vartheta s G_{xs} \right\} = 0,$$

where $G(T, x, s) = -\exp(-\alpha(x - h(s)))$.

Ansatz for value function: $G(t, x, s) = -\exp(-\alpha x) F(t, z)$ with $z = \ln(s)$

HJB equation becomes

$$\max_{\vartheta \in \mathbb{R}} \left\{ -F_t + \left( \vartheta (\mu_0 + \mu_1 \vartheta) \alpha - \frac{1}{2} \sigma^2 \vartheta^2 \alpha^2 \right) F + \left( \sigma^2 \vartheta \alpha + \frac{1}{2} \sigma^2 - \mu_0 - \mu_1 \vartheta \right) F_z - \frac{1}{2} \sigma^2 F_{zz} \right\} = 0,$$

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Holger Kraft, Christoph Kühn
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Holger Kraft, Christoph Kühn
Optimal strategy

\[ \hat{\theta}_t = \frac{\mu_0}{\alpha \sigma^2 - 2\mu_1} + \left(1 + \frac{\mu_1}{\alpha \sigma^2 - 2\mu_1}\right) \frac{F_Z(t, \ln(S_t))}{\alpha F(t, \ln(S_t))} \]

maximizer without claim

\[ =: \text{hedge multiplier} \]

\[ = \partial_z \rho^h(t, \ln(S_t)) \]

\[ \mu_1 = 0 \text{ (Black-Scholes)} \implies \text{hedge multiplier} = 1 \text{ (perfect hedging)} \]

\[ \mu_1 < 0 \implies \text{hedge multiplier} < 1 \text{ (underhedging)} \]

\[ \mu_1 > 0 \implies \text{hedge multiplier} > 1 \text{ (overhedging)} \]

Interpretation for the case \( \mu_1 < 0 \): large trader replicates e.g. 80% of the claim. The hedging portfolio suffers a loss from the price impact of the hedging activity (as price impact is negative). But the opposite derivative position profits from it. Taken together the 20% unhedged position profits from the price impact of 80% hedging activity.

Plugging the optimal stock position in the HJB-equation yields

\[ 0 = -F_t + \left(\mu_0 - \frac{1}{2} \sigma^2\right)F_Z - \frac{1}{2} \sigma^2 F_{zz} + \frac{1}{2} \left(\alpha \mu_0 F - \mu_1 F_Z + \sigma^2 \alpha F_Z^2\right) \frac{1}{\alpha (\alpha \sigma^2 - 2\mu_1) F} \]

Non linear!
Optimal strategy

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\( \mu_1 = 0 \) (Black-Scholes) \( \Rightarrow \) hedge multiplier \( = 1 \) (perfect hedging)
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Holger Kraft, Christoph Kühn
Optimal strategy

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$\mu_1 = 0$ (Black-Scholes) $\implies$ hedge multiplier $= 1$ (perfect hedging)

$\mu_1 < 0$ $\implies$ hedge multiplier $< 1$ (underhedging)

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Non linear!
Solution of the HJB-equation

To knock out the nonlinear term we use a trick applied in papers by Henderson, Hobson, and Zariphopoulou

Ansatz: \( F(t, z) = g(t, z)^\beta \)

and thus \( g(T, z) = \exp \left( \frac{\alpha}{\beta} h(\exp(z)) \right) \).

The HJB-equation becomes

\[
0 = -\frac{\beta}{\alpha} g_t - \frac{\gamma}{\alpha} (\mu_0 - \frac{1}{2} \sigma^2) g_z - \frac{1}{2} \frac{\gamma}{\alpha} \sigma^2 [ (\beta - 1) \frac{g_z^2}{g} + g_{zz} ] \\
+ \frac{1}{2} \frac{(\mu_0 g - \frac{\gamma}{\alpha} \mu_1 g_z + \beta \sigma^2 g_z)^2}{(\alpha \sigma^2 - 2 \mu_1) g}
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To knock out the terms with \( \frac{g_z^2}{g} \) we choose

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\beta = \frac{1}{1 - \frac{(\sigma^2 - \mu_1 / \alpha)^2}{\sigma^2 (\sigma^2 - 2 \mu_1 / \alpha)}} < 0
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\]
Solution of the HJB-equation

To knock out the nonlinear term we use a trick applied in papers by Henderson, Hobson, and Zariphopoulou

Ansatz: \[ F(t, z) = g(t, z)^{\beta} \]

and thus \( g(T, z) = \exp\left(\frac{\alpha}{\beta} h(\exp(z))\right) \).

The HJB-equation becomes

\[
0 = -\frac{\beta}{\alpha} g_t - \frac{\gamma}{\alpha} (\mu_0 - \frac{1}{2} \sigma^2) g_z - \frac{1}{2} \frac{\gamma}{\alpha} \sigma^2 \left[(\beta - 1) \frac{g_z^2}{g} + g_{zz}\right] \\
+ \frac{1}{2} \left(\mu_0 g - \frac{\gamma}{\alpha} \mu_1 g_z + \beta \sigma^2 g_z\right)^2 \\
\quad \frac{1}{2} \frac{1}{(\alpha \sigma^2 - 2 \mu_1)} g
\]

To knock out the terms with \( \frac{g_z^2}{g} \) we choose

\[
\beta = \frac{1}{1 - \frac{(\sigma^2 - \mu_1 / \alpha)^2}{\sigma^2 (\sigma^2 - 2 \mu_1 / \alpha)}} < 0
\]
\[
g_t - \frac{1}{2} \alpha \frac{\mu_0^2}{\alpha \sigma^2 - 2\mu_1} g + \left( \mu_0 - \frac{1}{2} \sigma^2 - \frac{\mu_0 (\alpha \sigma^2 - \mu_1)}{\alpha \sigma^2 - 2\mu_1} \right) g_z + \frac{1}{2} \sigma^2 g_{zz} = 0.
\]

This PDE is linear and thus it possesses a Feynman-Kac stochastic representation

\[
g(t, z) = \exp(-\tilde{\tau} (T - t)) \tilde{E} \left[ \exp \left( \frac{\alpha}{\beta} h(\exp(Z_T)) \right) \right], \quad \text{wobei}
\]

\(Z_T\) is normally distributed with expectation \(\eta_Z \cdot (T - t)\) & variance \(\sigma^2 \cdot (T - t)\)

For the **seller's indifference price** this yields

\[
p^h = \frac{1}{\alpha \beta} \ln \left( \tilde{E} \left[ \exp \left( \frac{\alpha}{\beta} h(\exp(Z_T)) \right) \right] \right).
\]

As \(\beta < 0\) this would formally correspond to the exponential principles (under the artificial measure \(\tilde{\mathbb{P}}\)) with the artificial **negative risk aversion** \(\frac{\alpha}{\beta}\).

**Consequence:** many things turn around
\[ g_t - \frac{1}{2} \frac{\alpha}{\beta} \frac{\mu_0^2}{\alpha \sigma^2 - 2 \mu_1} g + \left( \mu_0 - \frac{1}{2} \sigma^2 - \frac{\mu_0 (\alpha \sigma^2 - \mu_1)}{\alpha \sigma^2 - 2 \mu_1} \right) g_z + \frac{1}{2} \sigma^2 g_{zz} = 0. \]

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Seller’s indifference price:

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with \( \beta < 0 \).

- Seller’s indifference price is concave (and not convex as in (in)complete frictionless markets)
- Every claim \( h \geq 0 \) has a finite seller’s indifference price (even if Black-Scholes replication costs and expectation w.r.t. \( P \) are infinite)
- Hedging manipulation strategy \( \rightarrow \theta^{\text{Black-Scholes}} \) if risk aversion \( \alpha \rightarrow \infty \)
  \( \Rightarrow \) indifference price \( \rightarrow p^{\text{BS}} \) for \( \alpha \rightarrow \infty \)
- \( \frac{p^{\lambda h}}{\lambda} \rightarrow \text{ess inf}_{s \in \mathbb{R}^+} h(s), \quad \lambda \rightarrow \infty \)
  where the essential infimum is taken w.r.t. the Lebesgue measure on \( \mathbb{R} \)
  i.e. indifference price (per share) tends to minimal possible payoff of the derivative if position size \( \lambda \) explodes
  In the case of call/put options \( \text{ess inf}_{s \in \mathbb{R}^+} h(s) = 0 \)
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\( \theta^i \) is the €-amount that the \( i \)-th trader invests in stocks \((i = 1, 2)\)

**Stock price dynamics:**

\[
dS_t = S_t \left( (\mu_0 + \mu_1 \theta_1^i + \mu_1 \theta_2^i) \, dt + \sigma \, dW_t \right)
\]

**\( i \)-th player’s liquid wealth reads**

\[
dX_t^i = \frac{\theta^i_t}{S_t} \, dS_t = \theta^i_t (\mu_0 + \mu_1 \theta_1^i + \mu_1 \theta_2^i) \, dt + \theta^i_t \sigma \, dW_t, \quad i = 1, 2.
\]

Both traders maximize expected utilities from terminal wealths w.r.t. \( u_i(Y) = E_P \left[ -\exp(-\alpha_i Y) \right] \), \( i = 1, 2 \), with possibly different \( \alpha_1, \alpha_2 > 0 \)
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Consider the case that the first trader holds a short and the second a long position in the same illiquid derivative with payoff \( h(S_T) \)

- \( i = 1 \) (issuer) \( G^1(t, x, s) = -\exp(-\alpha_1(x - h(s))) \)
- \( i = 2 \) (holder) \( G^2(t, x, s) = -\exp(-\alpha_2(x + h(s))) \)

**Result:** The game has the following **Nash equilibrium**:

\[
\theta^1_t = \tilde{\theta}^1_t + S_t v_s(t, S_t) \quad \text{and} \quad \theta^2_t = \tilde{\theta}^2_t - S_t v_s(t, S_t),
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where

\[
\tilde{\theta}^i = \frac{(\alpha_j \sigma^2 - \mu_1) \mu_0}{\alpha_1 \alpha_2 \sigma^4 - 2 \sigma^2 \mu_1 (\alpha_1 + \alpha_2) + 3 \mu_1^2}, \quad i = 1, 2, \quad j \neq i
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and \( v(t, s) \) is the Black-Scholes price of the claim \( h(S_T) \)

- \( \implies \) price impacts of \( S_t v_s(t, S_t) \) and \( -S_t v_s(t, S_t) \) completely compensate \( \implies \) indifference prices = Black-Scholes price
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Intuition: Why is

\[ \theta_1^t = \hat{\theta}_1^t + S_t \nu_s(t, S_t) \quad \text{and} \quad \theta_2^t = \hat{\theta}_2^t - S_t \nu_s(t, S_t), \]

a Nash equilibrium?

Start with \((\theta_1^1, \theta_2^1)\) and show that for neither of the traders there is an incentive to change his strategy.

- Both traders hedge the risk of the derivative completely away.
- In addition, the price impacts of the hedging strategies \(S_t \nu_s(t, S_t)\) and \(-S_t \nu_s(t, S_t)\) completely compensate.
- Thus the situation is exactly the same as without the derivative deal with Nash equilibrium \((\hat{\theta}_1^1, \hat{\theta}_2^1)\).
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Many thanks for your attention!