Efficient Pricing of CMS Spread Options
in a Stochastic Volatility Libor Market Model

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Outline

Part 1: CMS Spread Option Pricing in a SV-LMM
- General Problem
- Spread option formulas
- Numerical Results

Part 2: Extracting Correlations from the Market
- Correlation Parameterizations
- Calibration Example
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Motivation

- Standard approach: Calibration of volatilities to caps/swaptions. Forward rate correlations are usually estimated from historical data.

- Swaptions exhibit only weak dependence on correlations.

- Even historical estimation of forward rate correlation matrices not trivial (choice of yield curve model, time horizon, backward-looking)

- Goal: Augment set of calibration instruments with CMS spread options. \( \Rightarrow \) Consistent pricing of spread-related exotics.
Model framework

SV-LMM with time-dep. skews (Piterbarg, 2005)

\[
\begin{align*}
    dL_i(t) &= \ldots dt + [\beta_i(t)L_i(t) + (1 - \beta_i(t))L_i(0)] \sqrt{V(t)} \sigma_i(t) dW_i(t), \\
    dV(t) &= \kappa(1 - V(t)) dt + \xi \sqrt{V(t)} dZ(t) \\
    \langle dW_n(t), dW_m(t) \rangle &= \rho(t, T_n, T_m) dt \\
    \langle dW_n(t), dZ(t) \rangle &= 0
\end{align*}
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\end{align*}
\]

Swap-rate dynamics

For the swap rates

\[S_j = \sum w_i L_i\]

we have approximately

\[dS_j(t) = \ldots dt + [\beta_j(t)S_j(t) + (1 - \beta_j(t))S_j(0)] \sqrt{V(t)} \sigma_j(t) dW_j(t),\]

i.e., they are again of type 'displaced Heston'.
Problem to be solved

For the calibration to CMS spread options we must be able to rapidly calculate expectations of the form

$$\mathbb{E}[(S_1(T) - S_2(T) - K)^+]$$,

where $S_1$ and $S_2$ are for example a 10Y and a 2Y swap rate, respectively.

⇒ No Monte-Carlo!

Derive the 2-dim Fourier transform $\hat{F}$ of the payoff function

$$F(x_1, x_2) = (e^{x_1} - e^{x_2} - 1)^+$$

via complex $\Gamma$-function.

$\Rightarrow$ Price of spread option given as 2-dim inverse Fourier transform

$$\text{SprOpt}(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2 + i\varepsilon} e^{iuX_0'} \hat{F}(u_1, u_2) \Phi(u_1, u_2) d(u_1, u_2),$$

where $\Phi$ denotes char. function of $(\log S_1(T), \log S_2(T))$.

However: Efficient 2-dim Fourier inversion not trivial! (Choice of $\varepsilon$, highly oscillatory integrand for short maturities)
Method II: Kiesel & Lutz (2010)

Brownian motion driving stochastic volatility is independent of the swap rate Brownian motions!

Define the integrated variance

\[ \overline{V}_T := \int_0^T V_t dt. \]

\[ \Rightarrow (S_1(T), S_2(T)) \mid \overline{V}_T \text{ jointly log-normal (if skews } = 1). \]

\[ \Rightarrow \mathbb{E}[(S_1(T) - S_2(T) - K)^+] = \mathbb{E} \left[ \mathbb{E}[(S_1(T) - S_2(T) - K)^+ \mid \overline{V}_T] \right] \]

\[ = \int_0^\infty \int_{-\infty}^{\infty} g(u, v) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du f(v) dv, \]

where \( g(u, v) \) is some ‘nice’ real function and \( f \) the density of the integrated variance \( \overline{V}_T \).
Inverse transform

Problem: Density $f$ of $V_T$ only given via Fourier inversion $\sim$ a further integral!

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{x(a+iu)} \hat{f}(a+iu) du.$$ 

$\Rightarrow$ For each $x$ at which the density is to be evaluated we need to calculate an oscillatory integral.
Inverse transform II

Real part of the integrand in the complex plane

Green line: 'optimal' linear contour.
Inverse transform III

With optimal linear contour and after transformation to finite integration interval we have

\[ f(x) = -\frac{1}{\pi} \int_0^1 \text{Im} \left( e^{x\tilde{s}(u)} \hat{f}(\tilde{s}(u)) \tilde{s}'(u) \right) du \]

⇒ With an adaptive integration scheme usually 20-40 sampling points are sufficient for most practical applications.
Remarks

We can now rapidly calculate spread option prices for two correlated displaced Heston processes.

For application within LMM we further need

▶ appropriate approximations for swap rate drift terms (convexity adjustments)

▶ replace time-dep. swap-rate parameters with constants via parameter averaging (see Piterbarg (2005)).

→ see Kiesel & Lutz 2010.
Numerical Results

fwd rate volatilities $\sigma_i(T-t)$

- 20Y LMM based on 6M rates, 5 factors
- Typical upward sloping initial yield curve (from 3% to 5%)
- Skew parameters: linearly decreasing from 90% to 40%
- SV parameters: $\kappa = 15\%$, $\xi = 130\%$
Numerical Results

Figure: Implied normal spread volatilities in basis points for 10Y-2Y CMS spread options with 5Y maturity (top) and 10Y maturity (bottom). Computing time $\sim 40$ ms.
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Correlation parameterizations

- **Schoenmakers-Coffey (2003) 2-parametric form:**
  \[ \rho_{ij} = \exp\left(-\frac{|i-j|}{N-1} \left(-\log \rho_\infty + \eta h(i,j)\right)\right), \]
  \[ \rho_\infty \in (0, 1), \quad \eta \in [0, -\log \rho_\infty] \]

- **Rebonato (2004) 3-parametric form (not necessarily pos. def.)**
  \[ \rho_{ij} = \rho_\infty + (1-\rho_\infty) \exp\left(-\beta |i-j| \exp\{-\alpha \min(i,j)\}\right), \]
  \[ \beta > 0, \quad \alpha \in \mathbb{R}, \quad \rho_\infty \in [0, 1) \]

- **Lutz (2010) 5-parametric form:**
  \[ \rho_{ij} = \rho_\infty + (1-\rho_\infty) \left[ \exp\left(-\beta (i^\alpha + j^\alpha)\right) + \vartheta_{ij}(\delta, \gamma) \sqrt{(1-\exp\{-2\beta i^\alpha\})(1-\exp\{-2\beta j^\alpha\})} \right] \]
  \[ \alpha, \beta > 0, \quad \gamma, \delta \in \mathbb{R}, \quad \rho_\infty \in [0, 1). \]
Lutz (2010) 5-parametric form
Calibration example

- 30Y semi-annual LMM, 10 factors.
- Piecewise const. forward rate volatility and skew functions.
- EUR market data as of 01/14/2008.
- Calibration targets: 9 caplet smiles, 36 swaption smiles, 7 CMS spread option (10Y-2Y) implied volatilities.
- Computing time: 1:54 min.

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<tr>
<th>RMSE</th>
<th>ATM</th>
<th>Smile</th>
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<tbody>
<tr>
<td>caplets (Black-vol)</td>
<td>0.24%</td>
<td>0.48%</td>
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<tr>
<td>swaptions (Black-vol)</td>
<td>0.25%</td>
<td>0.31%</td>
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<table>
<thead>
<tr>
<th></th>
<th>K = 0.25%</th>
<th>Smile</th>
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<tbody>
<tr>
<td>CMSSOs (bp vol)</td>
<td>0.7</td>
<td>2.4</td>
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**Calibration example**

Figure: 10Y-2Y CMS spread option implied volatilities for $K = 0.25\%$. 
Calibration example

Figure: Calibrated correlation matrices. From top left to bottom right: 5P fitted to historical matrix, SC2, Reb3, 5P.
References

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Thank you for your attention !