Dynamic Hedging of Conditional Value-at-Risk
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In this talk, the problem of partial hedging is studied by constructing hedging strategies that minimize conditional value-at-risk (CVaR) of the portfolio. Two aspects of the problem are considered: minimization of CVaR with initial capital bounded from above, and minimization of hedging costs subject to a CVaR constraint. The Neyman-Pearson lemma is used to deduce semi-explicit solutions. The results are illustrated by constructing CVaR-efficient hedging strategies for a call option in the Black-Scholes model, call option in regime-switching telegraph market model and embedded call option for equity-linked life insurance contract.
In a complete unconstrained financial market every contingent claim with discounted payoff $H$ can be hedged perfectly.

Perfect hedging requires initial capital in the amount of $H_0 = \mathbb{E}_{\mathbb{P}^*}[H]$.

In a constrained market perfect hedging is not always possible.

Example of a constraint: initial capital bounded by $\tilde{V}_0 < H_0$.

The problem is to select the “best” partial hedging strategy.

One of the approaches is to optimize a risk measure.
Properties of the optimal hedging strategy depend on the risk measure being optimized.

Poor choice of the risk measure generally leads to poor results.

Examples of risk measures:

- Linear shortfall risk
- Quadratic loss
- Probability of successful hedging
- Value-at-risk
- Conditional value-at-risk
- Lower/upper tail conditional expectation
- Worst conditional expectation
- Expected shortfall
Let random variable $L$ represent loss (can be negative).

- **Linear shortfall risk**: $\mathbb{E}_P[L^+]$, where $x^+ = \max(x, 0)$.
- **Quadratic loss**: $\mathbb{E}_P[L^2]$.
- **Probability of successful hedging**: $\mathbb{P}(L \leq 0)$. 

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**Dynamic Hedging of Conditional Value-at-Risk**
Choosing a Risk Measure
Value-at-Risk and Conditional Value-at-Risk

- VaR and CVaR are defined for a fixed level \( \alpha \in (0, 1) \).
- Let \( L(\alpha) \) and \( L^{(\alpha)} \) be lower and upper \( \alpha \)-quantiles of \( L \):
  \[
  L(\alpha) = \inf\{x \in \mathbb{R} : \mathbb{P}[L \leq x] \geq \alpha\},
  \]
  \[
  L^{(\alpha)} = \inf\{x \in \mathbb{R} : \mathbb{P}[L \leq x] > \alpha\}
  \]
- **Value-at-risk (VaR)** at level \( \alpha \):
  \[
  \text{VaR}^\alpha(L) = L^{(1-\alpha)}.
  \]
- **Conditional value-at-risk (CVaR)** at level \( \alpha \):
  \[
  \text{CVaR}^\alpha(L) = \inf \left\{ z + \frac{1}{\alpha} \cdot \mathbb{E}_\mathbb{P} \left[ (L - z)^+ \right] : z \in \mathbb{R} \right\}.
  \]
- Note that the infimum in CVaR definition is always attained as minimum (see Rockafellar and Uryasev, 2000).
Choosing a Risk Measure
Tail Conditional Expectation, Worst Conditional Expectation and Expected Shortfall

- **Lower tail conditional expectation (lower TCE) at level** $\alpha$:
  \[
  \text{TCE}_\alpha(L) = \mathbb{E}[L \mid L \geq L(1-\alpha)].
  \]

- **Upper tail conditional expectation (upper TCE) at level** $\alpha$:
  \[
  \text{TCE}_\alpha(L) = \mathbb{E}[L \mid L \geq L^{(1-\alpha)}].
  \]

- **Worst conditional expectation (WCE) at level** $\alpha$:
  \[
  \text{WCE}_\alpha(L) = \sup \{ \mathbb{E}[L \mid A] : A \in \mathcal{F}, \mathbb{P}[A] > \alpha \}.
  \]

- **Expected shortfall (ES) at level** $\alpha$:
  \[
  \text{ES}_\alpha(L) = \frac{1}{\alpha} \cdot \left( \mathbb{E}[L \cdot 1_{\{L \geq L(1-\alpha)\}}] + L(1-\alpha) \cdot \left( \mathbb{P}[L \geq L(1-\alpha)] - \alpha \right) \right).
  \]
The following relationships are true for any loss function:

\[
\begin{align*}
\text{ES}_\alpha &= \text{CVaR}_\alpha, \\
\text{TCE}_\alpha &\leq \text{TCE}_\alpha \leq \text{CVaR}_\alpha, \\
\text{TCE}_\alpha &\leq \text{WCE}_\alpha \leq \text{CVaR}_\alpha.
\end{align*}
\]

\[
\text{TCE}_\alpha(L) = \text{TCE}_\alpha(L) = \text{WCE}_\alpha(L) = \text{CVaR}_\alpha(L) \text{ if and only if}
\]

\[
\mathbb{P}(L \geq L^{(1-\alpha)}) = \alpha, \mathbb{P}(L > L^{(1-\alpha)}) > 0
\]

or

\[
\mathbb{P}(L \geq L^{(1-\alpha)}, L \neq L^{(1-\alpha)}) = 0.
\]
Consider a world with three states: \( P(\omega_1) = P(\omega_2) = 0.48, \ P(\omega_3) = 0.04 \) and three different loss functions: \( L_1, L_2 \) and \( L_3 \).

<table>
<thead>
<tr>
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<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
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<th>( \text{VaR}_{0.05} )</th>
<th>( \mathbb{E}[L^2] )</th>
<th>( \text{CVaR}_{0.05} )</th>
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<tr>
<td>( L_1 )</td>
<td>-1</td>
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<td>( L_2 )</td>
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<td>0.48</td>
<td>1.00</td>
<td>6.40</td>
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In the example above:

- \( P[L_1 \leq 0] = P[L_2 \leq 0] = P[L_3 \leq 0] \),
- \( \text{VaR}_{0.05}(L_1) = \text{VaR}_{0.05}(L_2) = \text{VaR}_{0.05}(L_3) \),
- \( \mathbb{E}[(L_1)^2] \leq \mathbb{E}[(L_3)^2] \leq \mathbb{E}[(L_2)^2] \),
- \( \text{CVaR}_{0.05}(L_1) = \text{CVaR}_{0.05}(L_3) \leq \text{CVaR}_{0.05}(L_2) \).
Let the discounted price process $X_t$ be a semimartingale on a standard stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

A self-financing strategy: initial capital $V_0 > 0$ and a predictable process $\xi_t$. For each strategy $(V_0, \xi)$ the value process $V_t$ is

$$V_t = V_0 + \int_0^t \xi_s dX_s, \quad \forall t \in [0, T].$$

A strategy $(V_0, \xi)$ is admissible if

$$V_t \geq 0, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.$$

Denote the set of all admissible self-financing strategies by $\mathcal{A}$. 
Consider a short position in a contingent claim whose discounted payoff is an $\mathcal{F}_T$–measurable random variable $H \in L^1(\mathbb{P})$, $H \geq 0$.

In a complete market there exists a unique martingale measure $\mathbb{P}^* \approx \mathbb{P}$, and the perfect hedging strategy requires allocating initial capital $H_0 = \mathbb{E}_{\mathbb{P}^*}[H]$ (risk-neutral price).

For each strategy $(V_0, \xi)$ define loss function:

$$L = L(V_0, \xi) = H - V_T.$$

Capital constraint: $V_0 \leq \tilde{V}_0 < H_0$.

The problem is to minimize CVaR over the set of admissible self-financing strategies:

$$\begin{cases} 
\text{CVaR}_\alpha(V_0, \xi) \longrightarrow \min_{(V_0, \xi) \in \mathcal{A}}, \\
V_0 \leq \tilde{V}_0.
\end{cases}$$
Recall that

$$\text{CVaR}^\alpha (V_0, \xi) = \inf \left\{ z + \frac{1}{\alpha} \cdot E_P \left[ (H - V_T - z)^+ \right] : z \in \mathbb{R} \right\},$$

and define

$$\mathcal{A}_{\tilde{V}_0} = \{(V_0, \xi) \mid (V_0, \xi) \in \mathcal{A}, \ V_0 \leq \tilde{V}_0 \},$$

$$c(z) = z + \frac{1}{\alpha} \cdot \min_{(V_0, \xi) \in \mathcal{A}_{\tilde{V}_0}} E_P \left[ (H - V_T - z)^+ \right].$$

Then

$$\min_{(V_0, \xi) \in \mathcal{A}_{\tilde{V}_0}} \text{CVaR}^\alpha (V_0, \xi) = \min_{z \in \mathbb{R}} c(z).$$

If we manage to derive an explicit form for $c(z)$, the initial problem is reduced to a problem of one-dimensional minimization.
The problem is to find an explicit expression for the function

\[ c(z) = z + \frac{1}{\alpha} \cdot \min_{(V_0, \xi) \in \tilde{A}_V_0} \mathbb{E}_P \left[ (H - V_T - z)^+ \right]. \]

Note that \((H - V_T - z)^+ \equiv ((H - z)^+ - V_T)^+\) and consider the problem

\[ \mathbb{E}_P \left[ (H - z)^+ - V_T \right]^\pm \longrightarrow \min_{(V_0, \xi) \in \tilde{A}_V_0} . \]

The latter is a problem of linear shortfall risk minimization with respect to a contingent claim whose payoff \((H - z)^+\) depends on parameter \(z\). The solution \((\hat{V}_0(z), \hat{\xi}(z))\) may be derived with the help of Neyman-Pearson lemma (Föllmer and Leukert, 2000).
The optimal strategy \((\hat{V}_0(z), \hat{\xi}(z))\) for the problem

\[
\mathbb{E}_{\mathbb{P}} \left[ (H - z)^+ - V_T^+ \right] \rightarrow \min_{(V_0, \xi) \in \mathcal{A}_{\tilde{V}_0}}
\]

is a perfect hedge for \(\tilde{H}(z) = (H - z)^+ \tilde{\varphi}(z)\), where

\[
\tilde{\varphi}(z) = \mathbbm{1}\left\{ \frac{d\mathbb{P}}{d\mathbb{P}^*} > \tilde{a}(z) \right\} + \gamma(z) \cdot \mathbbm{1}\left\{ \frac{d\mathbb{P}}{d\mathbb{P}^*} = \tilde{a}(z) \right\},
\]

\[
\tilde{a}(z) = \inf \left\{ a \geq 0 : \mathbb{E}_{\mathbb{P}^*} \left[ (H - z)^+ \cdot \mathbbm{1}\left\{ \frac{d\mathbb{P}}{d\mathbb{P}^*} > a \right\} \right] \leq \tilde{V}_0 \right\},
\]

\[
\gamma(z) = \frac{\tilde{V}_0 - \mathbb{E}_{\mathbb{P}^*} \left[ (H - z)^+ \cdot \mathbbm{1}\left\{ \frac{d\mathbb{P}}{d\mathbb{P}^*} > \tilde{a}(z) \right\} \right]}{\mathbb{E}_{\mathbb{P}^*} \left[ (H - z)^+ \cdot \mathbbm{1}\left\{ \frac{d\mathbb{P}}{d\mathbb{P}^*} = \tilde{a}(z) \right\} \right]}.
\]
The optimal strategy \((\hat{V}_0, \hat{\xi})\) for the problem

\[
\text{CVaR}_\alpha(V_0, \xi) \longrightarrow \min_{(V_0, \xi) \in A} \tilde{V}_0
\]

is a perfect hedge for \(\tilde{H}(\hat{z}) = (H - \hat{z})^+ \tilde{\phi}(\hat{z})\), where \(\tilde{\phi}(z)\) is the randomized test from linear shortfall risk subproblem, \(\hat{z}\) is the point of global minimum of

\[
c(z) = \begin{cases} 
  z + \frac{1}{\alpha} \cdot E_P [(H - z)^+(1 - \tilde{\phi}(z))], & \text{for } z < z^*, \\
  z, & \text{for } z \geq z^*,
\end{cases}
\]

on interval \(z < z^*\), and \(z^*\) is a real root of equation

\[
\tilde{V}_0 = E_P^* [(H - z^*)^+].
\]

Besides, one always has

\[
\text{CVaR}_\alpha(\hat{V}_0, \hat{\xi}) = c(\hat{z}).
\]
The dual problem is to minimize initial capital subject to a CVaR constraint:

\[
\begin{align*}
V_0 & \rightarrow \min_{(V_0, \xi) \in A}, \\
\text{CVaR}_\alpha(V_0, \xi) & \leq \tilde{C}.
\end{align*}
\]

\[\iff\]

\[
\begin{align*}
\mathbb{E}_{P^*}[V_T] & \rightarrow \min_{V_T \in F_T}, \\
\text{CVaR}_\alpha(V_T) & \leq \tilde{C}.
\end{align*}
\]

Recall that

\[
\text{CVaR}_\alpha(V_0, \xi) = \min_{z \in \mathbb{R}} \left( z + \frac{1}{\alpha} \cdot \mathbb{E}_P(H - V_T - z)^+ \right)
\]

and consider a family of problems

\[
\begin{align*}
\mathbb{E}_{P^*}[V_T] & \rightarrow \min_{V_T \in F_T}, \\
\mathbb{E}_P(H - V_T - z)^+ & \leq (\tilde{C} - z) \cdot \alpha.
\end{align*}
\]
Lemma

Let $\tilde{x}$ be a solution of

$$\begin{cases} 
  f(x) \rightarrow \min, \\
  \min_{x \in X} g(x, z) \leq c.
\end{cases}$$

Then the following family of problems also admits solutions, denoted $\tilde{x}(z)$:

$$\begin{cases} 
  f(x) \rightarrow \min, \\
  g(x, z) \leq c.
\end{cases}$$

Besides, one always has

$$\tilde{x} = \tilde{x}(\tilde{z}),$$

where $z$ is a point of global minimum of $f(\tilde{x}(z))$. 

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Dynamic Hedging of Conditional Value-at-Risk
Let $\tilde{V}_T(z)$ be the solution of

\[
\begin{align*}
\mathbb{E}_{P^*}[V_T] &\longrightarrow \min_{\nu_T \in \mathcal{F}_T}, \\
\mathbb{E}_P(H - V_T - z)^+ &\leq (\tilde{C} - z) \cdot \alpha.
\end{align*}
\]

Then the solution of the dual problem

\[
\begin{align*}
\mathbb{E}_{P^*}[V_T] &\longrightarrow \min_{\nu_T \in \mathcal{F}_T}, \\
\text{CVaR}_\alpha(V_T) &\leq \tilde{C}.
\end{align*}
\]

can be expressed as $\tilde{V}_T = \tilde{V}_T(\tilde{z})$, where

\[
\mathbb{E}_{P^*}[\tilde{V}_T(\tilde{z})] = \min_{z \in \mathbb{R}} \mathbb{E}_{P^*}[\tilde{V}_T(z)].
\]
If $\mathbb{E}_\mathbb{P}[H] > \tilde{C}\alpha$ and $\mathbb{E}_\mathbb{P}[(H - \tilde{C})^+] > 0$, the optimal strategy $(\hat{V}_0, \hat{\xi})$ for the dual problem is a perfect hedge for $(H - \hat{z})^+ (1 - \tilde{\phi}(\hat{z}))$, where $\tilde{\phi}(z)$ is defined by

$$
\tilde{\phi}(z) = 1 \left\{ \frac{d\mathbb{P}^*}{d\mathbb{P}} > \tilde{a}(z) \right\} + \gamma(z) \cdot 1 \left\{ \frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{a}(z) \right\},
$$

$$
\tilde{a}(z) = \inf \left\{ a \geq 0 : \mathbb{E}_\mathbb{P} \left[ (H - z)^+ \cdot 1 \left\{ \frac{d\mathbb{P}^*}{d\mathbb{P}} > a \right\} \right] \leq (\tilde{C} - z)\alpha \right\},
$$

$$
\gamma(z) = \frac{(\tilde{C} - z)\alpha - \mathbb{E}_\mathbb{P} \left[ (H - z)^+ \cdot 1 \left\{ \frac{d\mathbb{P}^*}{d\mathbb{P}} > \tilde{a}(z) \right\} \right]}{\mathbb{E}_\mathbb{P} \left[ (H - z)^+ \cdot 1 \left\{ \frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{a}(z) \right\} \right]},
$$

and $\hat{z}$ is a point of minimum of function

$$
d(z) = \mathbb{E}_{\mathbb{P}^*} \left[ (H - z)^+ (1 - \tilde{\phi}(z)) \right]
$$

on interval $-\infty < z \leq \tilde{C}$. 
Minimizing Hedging Costs
Dual Problem: Final Results (Part 2)

- If \( E_P[H] \leq \tilde{C} \alpha \) or \( E_P[(H - \tilde{C})^+] \leq 0 \), the optimal strategy \((\hat{V}_0, \hat{\xi})\) for the dual problem is a passive strategy (do nothing).
- If the first inequality is not satisfied, target CVaR is too high compared to the expected payoff on the contingent claim, so there is no need to hedge.
- If the second inequality is not satisfied, the payoff is bounded from above by a constant less than \( \tilde{C} \), so CVaR can never reach its target value no matter what strategy is used.
CVaR Hedging in the Black-Scholes Model

The Discounted Price Process

- Let the underlying $S_t$ and bond price $B_t$ follow
  \[
  \begin{align*}
  B_t &= e^{rt}, \\
  S_t &= S_0 \exp(\sigma W_t + \mu t).
  \end{align*}
  \]

- SDE for the discounted price process $X_t = B_t^{-1} S_t$:
  \[
  \begin{align*}
  dX_t &= X_t (\sigma dW_t + m dt), \\
  X_0 &= x_0,
  \end{align*}
  \]
  where $m = \mu - r + \frac{\sigma^2}{2}$.

- Terminal value and Radon-Nikodym derivative:
  \[
  \begin{align*}
  X_T &= x_0 \exp \left( \sigma W_T + (m - \frac{1}{2} \sigma^2) T \right), \\
  \frac{d\mathbb{P}^*}{d\mathbb{P}} &= \exp \left( -\frac{m}{\sigma} W_T - \frac{1}{2} \left( \frac{m}{\sigma} \right)^2 T \right) = \text{const} \cdot X_T^{-m/\sigma^2}.
  \end{align*}
  \]
The contingent claim of interest is a plain vanilla call option with payoff \((S_T - K)^+\).

The discounted payoff \(H\) is equal to

\[ H = (X_T - Ke^{-rT})^+. \]

The initial capital \(H_0\) required for a perfect hedge is

\[ H_0 = \mathbb{E}_{\mathbb{P}^*}[H] = x_0 \Phi_+(Ke^{-rT}) - Ke^{-rT} \Phi_-(Ke^{-rT}), \]

where

\[ \Phi_{\pm}(K) = \Phi \left( \frac{\ln x_0 - \ln K}{\sigma \sqrt{T}} \pm \frac{1}{2} \sigma \sqrt{T} \right), \]

and \(\Phi(\cdot)\) is a c.d.f. for standard normal distribution.
CVaR Hedging in the Black-Scholes Model

Problem Setting

- Assume the initial capital $V_0$ is limited by $\tilde{V}_0 < H_0$.
- For simplicity of presentation, assume $m > 0$.
- Our goal is to derive a hedging strategy that minimizes CVaR of the portfolio.
The optimal strategy \((\hat{V}_0, \hat{\xi})\) is a perfect hedge for 
\[
\tilde{H}(\hat{z}) = (X_T - (Ke^{-rT} + \hat{z}))^+ \cdot 1_{\{X_T > \tilde{b}(\hat{z})\}},
\]
where \(\hat{z}\) is a point of global minimum of \(c(z)\) on \((-\infty, z^*)\),
\[
c(z) = z + \frac{1}{\alpha} \cdot x_0 e^{mT} \tilde{\Phi}_\pm \left(Ke^{-rT} + z\right) - \tilde{\Phi}_\pm(\tilde{b}(z))
\]
\[
- (Ke^{-rT} + z) \left[\tilde{\Phi}_\pm \left(Ke^{-rT} + z\right) - \tilde{\Phi}_\pm(\tilde{b}(z))\right],
\]
where \(\tilde{\Phi}_\pm(x) = \Phi_\pm(xe^{-mT})\), \(z^*\) is the solution of
\[
\tilde{V}_0 = x_0 \Phi_+(Ke^{-rT} + z^*) - (Ke^{-rT} + z^*) \Phi_-(Ke^{-rT} + z^*),
\]
and for each \(z \in \mathbb{R}\), \(\tilde{b}(z)\) is the solution of
\[
\begin{cases}
x_0 \Phi_+(b) - ((Ke^{-rT} + z)) \Phi_-(b) = \tilde{V}_0, \\
b \geq (Ke^{-rT} + z).
\end{cases}
\]
Consider a plain vanilla call option with strike price of $K = 110$ and time to maturity $T = 0.25$.

Assume that financial market evolves according to the Black-Scholes model with parameters

$$\sigma = 0.3, \quad \mu = 0.09, \quad r = 0.05.$$ 

Initial stock price is $S_0 = 100$.

The objective is to construct CVaR$_{0.025}$-optimal partial hedging strategies for the call option with variable amount of initial capital available, ranging from 0 to the fair price of the option.
CVaR Hedging in the Black-Scholes Model

Numerical Example: Optimal CVaR vs. Initial Capital (2)

Available Capital as Fraction of Fair Price

Optimal CVaR

Dynamic Hedging of Conditional Value-at-Risk
(Ω, ℱ, ℙ) is "financial" probability space, as described earlier.
Consider "actuarial" probability space (˜Ω, ˜ℱ, ˜ℙ).
Let random variable \( T(x) \) denote the remaining lifetime of a person aged \( x \).
Let \( T \rho_x = \tilde{ℙ}[T(x) > T] \) be a survival probability for the next \( T \) years of the insured.
Assume that \( T(x) \) does not depend on the evolution of financial market.
Insurance company is obliged to pay the benefit in the amount of $\bar{H}$ to the insured, giving the insured is alive at time $T$.

$\bar{H}$ is an $\mathcal{F}_T$-measurable non-negative random variable.

The optimal price is traditionally calculated as an expected present value of cash flows under the risk-neutral probability.

The ”insurance” part of the contract doesn’t need to be risk-adjusted since the mortality risk is unsystematic.

Brennan-Shwartz price of the contract:

$$
\tau U_x = \mathbb{E}_\mathbb{P} \left\{ \mathbb{E}_{\mathbb{P}^*} \left[ H \cdot 1_{\{T(x) > T\}} \right] \right\} = \tau p_x \cdot \mathbb{E}_{\mathbb{P}^*} \left[ H \right],
$$

where $H = \bar{H}e^{-rT}$ is the discounted benefit.
The problem of the insurance company is to mitigate financial part of risk and hedge \( \bar{H} \) in the financial market.

However,

\[
\tau U_x < \mathbb{E}_{\mathbb{P}^*} [H],
\]

hence the perfect hedge is not accessible.

For a fixed client age \( x \) and time horizon \( T \), denote

\[
\tilde{V}_0 = \tau p_x \cdot \mathbb{E}_{\mathbb{P}^*} [H].
\]

We can now consider the problem of CVaR-optimal hedging of \( \bar{H} \) with capital constraint \( V_0 \leq \tilde{V}_0 \) and apply all techniques described earlier to derive the solution.

The related dual problem can also be considered.
Consider an equity-linked pure endowment contract with benefit being a call option with strike price of $K = 110$.

Let the starting price of the underlying be equal to $X_0 = 100$.

Let ”financial” world be driven by the Black-Scholes model:

$$\mu = 0.09, \quad r = 0.05, \quad \sigma = 0.3.$$ 

We optimize CVaR of hedging strategies for confidence level $\alpha = 0.025$ and variable time horizon.

We use survival probabilities from mortality table UP94 @ 2015 (Uninsured Pensioner Mortality 1994 Table Projected to the Year 2015) from McGill et al., ”Fundamentals of Private Pensions” (2004)).
CVaR Hedging of Equity-Linked Insurance Contracts

Numerical Example: Optimal CVaR for Ages 1-70

![Graph showing optimal CVaR for different ages and time periods. The graph includes lines for different time periods (T = 5, T = 10, T = 15, T = 20, T = 25) and shows how the optimal CVaR increases with age and time.]

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Dynamic Hedging of Conditional Value-at-Risk
Numerical Example: Optimal CVaR for Ages 1-35

The graph illustrates the optimal CVaR values for different client ages and time periods. The x-axis represents the client age, ranging from 0 to 35 years, and the y-axis represents the optimal CVaR values. For each time period (T), the graph shows how the optimal CVaR changes with age. The curves for different time periods are color-coded as follows:

- Blue: T = 5
- Green: T = 10
- Red: T = 15
- Cyan: T = 20
- Purple: T = 25

The optimal CVaR generally increases with age and time period, indicating a higher risk or variability in the outcomes for older clients and longer time periods.
Let \( \sigma(t) \in \{1, 2\} \), \( \sigma(0) = 1 \) be a continuous time Markov chain process with Markov generator

\[
L_\sigma = \begin{pmatrix}
-\lambda_1 & \lambda_1 \\
\lambda_2 & -\lambda_2
\end{pmatrix}.
\]

Define the main driving factors of the market:

\[
X_t = \int_0^t c_{\sigma(s)} ds, \quad J_t = \sum_{0}^{N_t} h_{\sigma(T_j-)},
\]

where \( c = (c_1, c_2) \), \( h = (h_1, h_2) \) and \( N_t \) is the number of jumps of \( \sigma(t) \) up to time \( t \).

The risk-free asset is defined by \( dB_t = r_{\sigma(t)} B_t dt \), and the interest rate \( r \) has two states \( (r_1, r_2) \).

The risky asset is defined similarly to Merton’s model:

\[
dS_t = S_t - d(X_t + J_t).
\]
Telegraph market model can be described as a complete market model with two traded assets, where dynamics of the risky asset features jumps and regime switching.

The model can be viewed as a generalization of Merton’s model preserving completeness of the market.

Theorem

The telegraph model is arbitrage free if and only if

\[
\frac{r_\sigma - c_\sigma}{h_\sigma} > 0, \quad \sigma = 1, 2.
\]

If the model is arbitrage free, it is complete.
Our algorithm for deriving CVaR-optimal strategies requires computing expectations of the form

$$\mathbb{E}[f(S_T, B_T) \cdot 1\{Z_T < a\}]$$

for various functions $f$ and constants $a$, both under the statistical measure $\mathbb{P}$ and under the risk-neutral measure $\mathbb{P}^*$. $S_t, B_t$ and $Z_t$ may all be expressed in terms of $X_t$ and $N_t$; consider $g(\cdot, \cdot)$ such that

$$\mathbb{E}[f(S_T, B_T) \cdot 1\{Z_T < a\}] = \mathbb{E}[g(X_t, N_t)].$$
Expand the expected value above by conditioning on $N_t = n$:

$$
\mathbb{E}[g(X_t, N_t)] = \sum_{n \geq 0} \int \mathbb{R} g(x, n) p_n(t, x) dx,
$$

where $p_n(t, x)$ is defined as

$$
p_n(t, x) = \frac{d}{dx} \mathbb{P} \left[ \{X_t < x\} \cap \{N_t = n\} \right].
$$
For all $t \geq 0$ and $x \in \mathbb{R}$,

$$p_0(t, x) = e^{-\lambda_1 t} \delta(x - c_1 t)$$

and for all $k \geq 1$

\[
p_{2k-1}(t, x) = \frac{\lambda_1 (\phi_1(t, x)\phi_2(t, x))^{k-1}}{|c_2 - c_1| ((k-1)!!)^2} \exp \left(-\phi_1(t, x) - \phi_2(t, x)\right),
\]

\[
p_{2k}(t, x) = \frac{p_{2k-1}(t, x)\phi_2(t, x)}{k},
\]

where

\[
\phi_1(t, x) = \lambda_1 \frac{c_2 t - x}{c_2 - c_1},
\]

\[
\phi_2(t, x) = \lambda_2 \frac{x - c_1 t}{c_2 - c_1},
\]

and $x \in (\min\{c_1 t, c_2 t\}, \max\{c_1 t, c_2 t\})$. 
Consider a plain vanilla call option with strike price of $K = 100$ and time to maturity $T = 1$.

Assume that financial market evolves according to the telegraph market model with parameters

\[
\begin{align*}
    c_1 &= -0.5, \quad c_2 = 0.5, \\
    \lambda_1 &= \lambda_2 = 5, \\
    r_1 &= r_2 = 0.07, \\
    h_1 &= 0.5, \quad h_2 = -0.35.
\end{align*}
\]

Initial stock price is $S_0 = 100$.

The objective is to construct CVaR_{0.025}-optimal partial hedging strategies for the call option with variable amount of initial capital available, ranging from 0 to the fair price of the option.