Multiscale Stochastic Volatility Models

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Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives

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Price Expansion

$P$: price of a vanilla European option (to start with)

\[
P = P_0 + v_0 \partial_{\sigma} P_0 + v_1 D_1 \partial_{\sigma} P_0 + v_2 D_2 P_0 + v_3 D_1 D_2 P_0 + v_4 \partial_{\sigma \sigma} P_0 + \cdots
\]

\[
D_1 = S \frac{\partial}{\partial S} (\text{Delta}), \quad D_2 = S^2 \frac{\partial^2}{\partial S^2} (\text{Gamma}) \quad \partial_{\sigma} = \frac{\partial}{\partial \sigma} (\text{Vega}) \cdots
\]

\[
v_i = v_i(\tau), \text{ payoff independent, } \quad \tau = \text{ time-to-maturity}
\]

$P_0$ is typically a constant volatility price → closed-form formula

Black-Scholes in Equity (Vasicek or CIR in Fixed Income, Black-Cox in Credit, ...)

Where do we get such an expansion?
What do we expect from it?
Wish List

\[ P = P_0 + v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + v_2 D_2 P_0 + v_3 D_1 D_2 P_0 + v_4 \partial_{\sigma\sigma} P_0 + \cdots \]

- **Accuracy**: the truncated expansion should be a good approximation \((v_i \to 0\) fast enough)
Wish List

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\[ + v_4 \partial^2_{\sigma\sigma} P_0 + \cdots \]

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- **Stability**: the coefficients \(v\)’s should be stable in time
  “short-time tight-fit vs. long-time rough fit”
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Let’s look at **calibration** first →
Calibration on Implied Volatilities

For vanilla European options we have: \( \partial_{\sigma} P_0 = \tau \bar{\sigma} D_2 P_0 \) so that

\[
P = P_0 + v_0 \partial_{\sigma} P_0 + v_1 D_1 \partial_{\sigma} P_0 + \frac{v_2}{\bar{\sigma}_T} \partial_{\sigma} P_0 + \frac{v_3}{\bar{\sigma}_T} D_1 \partial_{\sigma} P_0 + \cdots
\]
Calibration on Implied Volatilities

For vanilla European options we have: $\partial_\sigma P_0 = \tau \bar{\sigma} D_2 P_0$ so that

$$P = P_0 + v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + \frac{v_2}{\bar{\sigma} \tau} \partial_\sigma P_0 + \frac{v_3}{\bar{\sigma} \tau} D_1 \partial_\sigma P_0 + \cdots$$

For Calls, $P_0 = C_{BS}$ and by direct computation

$$P = C_{BS} + \left\{ v_0 + \frac{v_2}{\bar{\sigma} \tau} + \left( v_1 + \frac{v_3}{\bar{\sigma} \tau} \right) \left( 1 - \frac{d_1}{\bar{\sigma} \sqrt{\tau}} \right) \right\} \partial_\sigma C_{BS} + \cdots$$

where $d_1 = \frac{-LM + (\tau + \frac{1}{2} \bar{\sigma}^2) \tau}{\bar{\sigma} \sqrt{\tau}}$, and $LM \equiv \log(K/S)$
Calibration on Implied Volatilities

For vanilla European options we have: $\partial_{\sigma}P_0 = \tau\bar{\sigma}D_2P_0$ so that

$$P = P_0 + v_0\partial_{\sigma}P_0 + v_1D_1\partial_{\sigma}P_0 + \frac{v_2}{\bar{\sigma}\tau}\partial_{\sigma}P_0 + \frac{v_3}{\bar{\sigma}\tau}D_1\partial_{\sigma}P_0 + \cdots$$

For Calls, $P_0 = C_{BS}$ and by direct computation

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where $d_1 = \frac{-LM+(r+\frac{1}{2}\bar{\sigma}^2)\tau}{\bar{\sigma}\sqrt{\tau}}$, and $LM \equiv \log(K/S)$

Expanding the implied volatility $I = \bar{\sigma} + I_1 + \cdots \rightarrow$

$$P \equiv C_{BS}(\bar{\sigma} + I_1 + \cdots) = C_{BS} + I_1\partial_{\sigma}C_{BS} + \cdots$$

$$\implies I_1 = v_0 + \frac{v_2}{\bar{\sigma}\tau} + \left(v_1 + \frac{v_3}{\bar{\sigma}\tau}\right) \left(1 - \frac{d_1}{\bar{\sigma}\sqrt{\tau}}\right) + \cdots$$

**Affine in LMMR:** $I = b + a \frac{LM}{\tau} + (\text{quartic in } LM) + \cdots$

where the term structure of the $v$’s ($\tau$ dependence) is important.
Calibration Examples

Goal: fit

\[ I = b + a \frac{LM}{\tau} + \text{(quartic in LM)} + \cdots \]

to the observed implied volatility surface.

We typically fit the parameters \( a, b, \ldots \) by regressing in LMMR maturity-by-maturity, then we fit their dependence in \( \tau \).

We will see that our expansion leads to \( a, b \) which are affine in \( \tau \).

Some examples →
S&P 500 Implied Volatility data on June 5, 2003 and fits to the affine LMMR approximation for six different maturities.
S&P 500 Implied Volatility data on June 5, 2003 and fits to the two-scales asymptotic theory. The bottom (resp. top) figure shows the linear regression of $b$ (resp. $a$) with respect to time to maturity $\tau$. 
Higher Order Expansion

\[ I \sim \sum_{j=0}^{4} a_j(\tau) (LM)^j + \frac{1}{\tau} \Phi_t, \]
S&P 500 Implied Volatility data on June 5, 2003 and quartic fits to the asymptotic theory for four maturities.
Stochastic Volatility Models

Equity for instance.
Under physical measure:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t^{(0)}
\]

\[
\sigma_t = f(Y_t, Z_t, \cdots)
\]

\[
dY_t = \alpha(Y_t) dt + \beta(Y_t) dW_t^{(1)}
\]

\[
dZ_t = c(Z_t) dt + g(Z_t) dW_t^{(2)}
\]

\[
\cdots
\]

Volatility factors can be differentiated by their time scales
Multiscale Stochastic Volatility Models

\[ \sigma_t = f(Y_t, Z_t) \]

- \( Y_t \) is fast mean-reverting (ergodic on a fast time scale):
  \[ dY_t = \frac{1}{\varepsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(1)}, \quad 0 < \varepsilon \ll 1 \]
Multiscale Stochastic Volatility Models

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- **\( Y_t \) is fast mean-reverting** (ergodic on a fast time scale):
  \[
dY_t = \frac{1}{\varepsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(1)}, \quad 0 < \varepsilon \ll 1
  \]
- **\( Z_t \) is slowly varying**:
  \[
dZ_t = \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)}, \quad 0 < \delta \ll 1
  \]
Multiscale Stochastic Volatility Models

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**Separation of time scales:** \( \epsilon \ll T \ll 1/\delta \)

(assuming \( f \) continuous in \( z \)):

\[ \frac{1}{T} \int_0^T \sigma_t^2 dt = \frac{1}{T} \int_0^T f^2(Y_t, Z_t) dt \longrightarrow \langle f^2(\cdot, z) \rangle_{\Phi_Y} \]

**Local Effective Volatility:** \( \bar{\sigma}^2(z) \equiv \langle f^2(\cdot, z) \rangle_{\Phi_Y} \)

\[ P_0 = P_{BS}(\bar{\sigma}(z)) \]
Market Prices of Volatility Risk

Under the risk neutral measure $\mathbb{P}^*$ chosen by the market:

\[
\begin{align*}
    dS_t &= rS_t dt + f(Y_t, Z_t)S_t dW_t^{(0)*}, \\
    dY_t &= \left( \frac{1}{\varepsilon} \alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) \Lambda(Y_t, Z_t) \right) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(1)*}, \\
    dZ_t &= \left( \delta c(Z_t) - \sqrt{\delta} g(Z_t) \Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)*}.
\end{align*}
\]

\[d < W^{(0)*}, W^{(1)*} >_t = \rho_1 dt\]
\[d < W^{(0)*}, W^{(2)*} >_t = \rho_2 dt\]

\(\Lambda\) and \(\Gamma\): market prices of volatility risk
Pricing Equation

\[ P^{\varepsilon,\delta}(t, x, y, z) = \mathbb{E}^{*}\left\{ e^{-r(T-t)}h(S_T) \mid S_t = x, Y_t = y, Z_t = z \right\} \]

Feynman–Kac:

\[
\left( \frac{1}{\varepsilon} \mathcal{L}_Y + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{\rho_1,\Lambda} + \mathcal{L} + \sqrt{\delta} \mathcal{L}_{\rho_2,\Gamma} + \delta \mathcal{L}_Z + \sqrt{\frac{\delta}{\varepsilon}} \mathcal{L}_{\rho_{12}} \right) P^{\varepsilon,\delta} = 0
\]

\[ P^{\varepsilon,\delta}(T, x, y, z) = h(x) \]

with

\[
\mathcal{L} = \mathcal{L}_{BS}(f(y, z)) = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y, z) x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right)
\]
Regular-Singular Perturbations

\[ P^{\varepsilon,\delta} = \sum_{i,j} \varepsilon^{i/2} \delta^{j/2} P_{i,j} = P_0 + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \cdots \]

\[ \mathcal{L}_{BS}(\bar{\sigma}(z)) P_0 = 0, \quad P_0(T, x) = h(x) \implies P_0 = P_{BS}(\bar{\sigma}(z)) \]

\( P_0 \) is independent of \( y \) and \( z \) is a parameter.
Regular-Singular Perturbations

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\( P_0 \) is independent of \( y \) and \( z \) is a parameter.

\[ \mathcal{L}_{BS}(\bar{\sigma}(z)) \left( \sqrt{\varepsilon} P_{1,0} \right) + V_2^\varepsilon D_2 P_{BS} + V_3^\varepsilon D_1 D_2 P_{BS} = 0 \]
\[ \mathcal{L}_{BS}(\bar{\sigma}(z)) \left( \sqrt{\delta} P_{0,1} \right) + 2 \left( V_0^\delta \partial_{\sigma} P_{BS} + V_1^\delta D_1 \partial_{\sigma} P_{BS} \right) = 0 \]
\[ P_{1,0}(T, x) = P_{0,1}(T, x) = 0 \]

\( V_0^\delta \) and \( V_2^\varepsilon \) are volatility level adjustments due to \( \Gamma \) and \( \Lambda \) resp. \( V_1^\delta \) and \( V_3^\varepsilon \) are skew parameters proportional to \( \rho_2 \) and \( \rho_1 \) resp.
Regular-Singular Perturbations

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\[ P_0 \text{ is independent of } y \text{ and } z \text{ is a parameter.} \]

\[ \mathcal{L}_{BS}(\bar{\sigma}(z)) \left( \sqrt{\varepsilon} P_{1,0} \right) + V^\varepsilon_2 D_2 P_{BS} + V^\varepsilon_3 D_1 D_2 P_{BS} = 0 \]

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\[ P_{1,0}(T,x) = P_{0,1}(T,x) = 0 \]

\( V^\delta_0 \) and \( V^\varepsilon_2 \) are volatility level adjustments due to \( \Gamma \) and \( \Lambda \) resp.

\( V^\delta_1 \) and \( V^\varepsilon_3 \) are skew parameters proportional to \( \rho_2 \) and \( \rho_1 \) resp.

**Important:** these Black-Scholes equations will hold for exotic options with additional boundary conditions, but with the same group parameters \( V \)'s
Explicit formulas for Vanilla European Options

Notation: \( T - t = \tau \)

\[
\sqrt{\epsilon} P_{1,0} = \tau \left( V^\epsilon_2 D_2 P_{BS} + V^\epsilon_3 D_1 D_2 P_{BS} \right)
\]

easily checked by using \( \mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS} \)
Explicit formulas for Vanilla European Options

Notation: \( T - t = \tau \)

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\sqrt{\varepsilon} P_{1,0} = \tau \left( V_2^\varepsilon D_2 P_{BS} + V_3^\varepsilon D_1 D_2 P_{BS} \right)
\]
easily checked by using \( \mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS} \)

\[
\sqrt{\delta} P_{0,1} = \tau \left( V_0^\delta \partial_\sigma P_{BS} + V_1^\delta D_1 \partial_\sigma P_{BS} \right)
\]
easily checked by using first \( \partial P_{BS} = \tau \bar{\sigma} D_2 P_{BS} \) and then \( \mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS} \)
Explicit formulas for Vanilla European Options

Notation: $T - t = \tau$

$$\sqrt{\varepsilon} P_{1,0} = \tau \left( V_2^\varepsilon D_2 P_{BS} + V_3^\varepsilon D_1 D_2 P_{BS} \right)$$

easily checked by using $\mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS}$

$$\sqrt{\delta} P_{0,1} = \tau \left( V_0^\delta \partial_\sigma P_{BS} + V_1^\delta D_1 \partial_\sigma P_{BS} \right)$$

easily checked by using $\partial P_{BS} = \tau \bar{\sigma} D_2 P_{BS}$ and then $\mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS}$.

• Back to our expansion $\rightarrow$

$$P = P_0 + v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + v_2 D_2 P_0 + v_3 D_1 D_2 P_0 + \cdots$$

$$v_0 = \tau V_0^\delta, \quad v_1 = \tau V_1^\delta$$

$$v_2 = \tau V_2^\varepsilon, \quad v_3 = \tau V_3^\varepsilon$$

In terms of calibration to implied volatilities $\rightarrow$
Implied Volatility Calibration Formulas

\[
\bar{\sigma} + \frac{V_2}{\bar{\sigma}} + \frac{V_3}{2\bar{\sigma}}(1 - \frac{2r}{\bar{\sigma}^2}) + \tau \left( V_0 + \frac{V_1}{2}(1 - \frac{2r}{\bar{\sigma}^2}) \right) + \left( \frac{V_3}{\bar{\sigma}^3} + \tau \frac{V_1}{\bar{\sigma}^2} \right)
\]

LMMR

Either

- one estimates \( \bar{\sigma} \) from historical data (preferred for hedging where \( V_0 \) and \( V_2 \) do not appear), and then fitting maturity-by-maturity and regressing in \( \tau \), one gets:
  1. \( V_1 \) and \( V_3 \) from the slope \( a \)
  2. \( V_0 \) and \( V_2 \) from the intercept \( b \)
**Implied Volatility Calibration Formulas**

\[
\bar{\sigma} + \frac{V_2}{2\bar{\sigma}} + \frac{V_3}{2\bar{\sigma}}(1 - \frac{2r}{\bar{\sigma}^2}) + \tau \left( \frac{V_0}{2} \left(1 - \frac{2r}{\bar{\sigma}^2}\right) \right) + \left( \frac{V_3}{\bar{\sigma}^3} + \tau \frac{V_1}{\bar{\sigma}^2} \right) \quad \text{LMMR}
\]

intercept \( b \)

slope \( a \)

Either

- one estimates \( \bar{\sigma} \) from **historical data** (preferred for **hedging** where \( V_0 \) and \( V_2 \) do not appear), and then fitting maturity-by-maturity and regressing in \( \tau \), one gets:
  1. \( V_1 \) and \( V_3 \) from the slope \( a \)
  2. \( V_0 \) and \( V_2 \) from the intercept \( b \)

- or one uses the **adjusted effective volatility** \( \sigma^* \equiv \sqrt{\bar{\sigma}^2 + 2V_2} \)
  calibrated from **option data**, along with \( V_0 \), \( V_1 \), and \( V_3 \) (preferred for **pricing**):

\[
\sigma^* + \frac{V_3}{2\sigma^*}(1 - \frac{2r}{\sigma^*2}) + \tau \left( \frac{V_0}{2} \left(1 - \frac{2r}{\sigma^*2}\right) \right) + \left( \frac{V_3}{\sigma^3} + \tau \frac{V_1}{\sigma^2} \right) \quad \text{LMMR}
\]
Back to the Wish List: Accuracy

If the payoff function $h$ is smooth:

\[
P^{\varepsilon,\delta} = \left( P_0 + \sqrt{\varepsilon} P_{1,0} + \varepsilon P_{2,0} + \varepsilon^{3/2} P_{3,0} \right) + \sqrt{\delta} \left( P_{0,1} + \sqrt{\varepsilon} P_{1,1} + \varepsilon P_{2,1} \right) + R^{\varepsilon,\delta}
\]

\[
= \left( P_0 + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} \right) + O(\varepsilon + \delta) + R^{\varepsilon,\delta}
\]

then the residual $R^{\varepsilon,\delta}$ satisfies

\[
\mathcal{L}^{\varepsilon,\delta} R^{\varepsilon,\delta} = O(\varepsilon + \delta)
\]

\[
R^{\varepsilon,\delta}(T) = O(\varepsilon + \delta)
\]

and therefore $R^{\varepsilon,\delta} = O(\varepsilon + \delta)$.

If $h$ is non-smooth (call option in particular), then use a careful regularization.
Path-Dependent Derivatives (Barrier, Asian,...)

- Calibrate $\sigma^*$, $V_0$, $V_1$ and $V_3$ on the implied volatility surface
- Solve the corresponding problem with constant volatility $\sigma^*$
  \[ \implies P_0 = P_{BS}(\sigma^*) \]
- Use $V_0$, $V_1$ and $V_3$ to compute the source
  \[ 2 (V_0 \partial_\sigma P_{BS}^* + V_1 D_1 \partial_\sigma P_{BS}^*) + V_3 D_1 D_2 P_{BS}^* \]
- Get the correction by solving the SAME PROBLEM with zero boundary conditions and the source.
American Options

- Calibrate $\sigma^*$, $V_0$, $V_1$ and $V_3$ on the **implied volatility surface**
- Solve the corresponding problem with **constant volatility** $\sigma^*$
  \[
  \Rightarrow P^* \text{ and the free boundary } x^*(t)
  \]

- Use $V_0$, $V_1$ and $V_3$ to compute the **source**
  \[
  2 \left( V_0 \partial_\sigma P^*_{BS} + V_1 D_1 \partial_\sigma P^*_{BS} \right) + V_3 D_1 D_2 P^*_{BS}
  \]

- Get the **correction** by solving the corresponding problem with **fixed boundary** $x^*(t)$, **zero boundary conditions** and the **source**.
Cost of the Black-Scholes Hedging Strategy

\[
P_{BS}(T, S_T) = h(S_T)
\]

\[
P_{BS}(t, S_t) = a_t S_t + b_t e^{r t}, \quad a_t = \partial_x P_{BS}
\]

**Infinitesimal cost:**

\[
dP_{BS}(t, S_t) - \underbrace{(a_t dS_t + rb_t e^{r t} dt)}_{\text{self-financing part}} = \frac{1}{2} \left( f^2(Y_t, Z_t) - \sigma^2 \right) D_2 P_{BS}(t, S_t) dt
\]

**Cumulative financing cost:**

\[
E_{BS}(t) = \frac{1}{2} \int_0^t e^{-rs} \left( f^2(Y_s, Z_s) - \sigma^2 \right) D_2 P_{BS}(s, S_s) ds
\]

Choice of \( \sigma \)?
Choice of $\sigma$?

Since $Y_t$ is fast mean-reverting ($\varepsilon \ll 1$), integrals like

$$\int_0^t \left( f^2(Y_s, Z_s) - \sigma^2 \right) \Psi_s ds$$

will be small with $\varepsilon$ if

$$\sigma^2 = \bar{\sigma}^2(z) = \langle f^2(\cdot, z) \rangle \Phi(Y)$$

Therefore two choices:

- $\sigma^2 = \bar{\sigma}^2(Z_t)$ and $P_{BS} = P_{BS}(t, S_t; \bar{\sigma}(Z_t))$, in which case $\bar{\sigma}(Z_t)$ needs to be estimated continuously (and $dP_{BS}$ revisited)

- $\sigma^2 = \bar{\sigma}^2(Z_0)$ and $P_{BS} = P_{BS}(t, S_t; \bar{\sigma}(Z_0))$ with

$$f^2(Y_s, Z_s) - \sigma^2 = (f^2(Y_s, Z_s) - \bar{\sigma}^2(Z_t)) + (\bar{\sigma}^2(Z_t) - \bar{\sigma}^2(Z_0))$$

in which case parameters are frozen at time zero, an additional cost of order $\sqrt{\delta}$ comes from the second term (offset in practice by re-calibration at $\sqrt{\delta}$-frequency).
Corrected Hedging Strategy

A careful analysis of the cost shows

\[ E_0(t) = \frac{1}{2} \int_0^t e^{-rs} \left( f^2(Y_s, Z_s) - \bar{\sigma}^2(Z_t) \right) D_2 P_{BS}(s, S_s) ds \]

\[ = \sqrt{\varepsilon} (B^\varepsilon_t + M^\varepsilon_t) + O(\varepsilon + \delta), \]

where \( M^\varepsilon_t \) is a martingale, and

\[ B^\varepsilon_t = -\frac{\rho_1}{2} \int_0^t e^{-rs} \beta(Y_s) \frac{\partial \phi}{\partial y} f(Y_s, Z_s) D_1 D_2 P_{BS}(s, S_s) ds \]

is a bounded variation bias which can be compensated by using the corrected hedging ratio \( a_t \) given by

\[ \partial_x P_{BS} + (T - t)V_3 \partial_x D_1 D_2 P_{BS} + (T - t)V_1 \partial_x D_1 \partial_\sigma P_{BS} \]

The last term compensates for the bias generated by \( \bar{\sigma}^2(Z_t) - \bar{\sigma}^2(Z_0) \)
Examples of other:

- Models
- Regimes
- Applications
A Model with Volatility Time-Scale of Order One

In the model $\sigma_t = f(Y_t, Z_t)$, if one wants to:

- keep $Y$ fast mean-reverting
- let $Z$ be on a time scale comparable to maturity (or add one such factor)
- keep the computational tractability

then, one needs to make sure that the SV model $\bar{\sigma}^2(Z_t)$ is tractable.
A Model with Volatility Time-Scale of Order One

In the model $\sigma_t = f(Y_t, Z_t)$, if one wants to:

- keep $Y$ fast mean-reverting
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- keep the computational tractability

then, one needs to make sure that the SV model $\bar{\sigma}^2(Z_t)$ is tractable.

An interesting choice is the **Heston model**:
“A Fast Mean-Reverting Correction to Heston Stochastic Volatility Model” with **Matthew Lorig** (PhD student, UCSB), where we develop this idea.

An example of fit
SPX Implied Volatilities from May 17, 2006
Fast Mean-Reverting SV and Short Maturities

If the time scale of the fast mean-reverting factor $Y$ is $\varepsilon << 1$, and if the maturity of interest is small but still large compared with $\varepsilon$, then, one can consider the regime

$$\varepsilon << T \sim \sqrt{\varepsilon} << 1$$

It involves a non-trivial mixture of Large Deviation (short maturity) and Homogenization (fast mean reverting coefficient):


Interestingly, in this regime and for this model, we derive explicit formulas for the limiting implied volatility which looks like
Three parameters which control the implied volatility skew’s level ($\theta$), slope ($\rho$) and convexity ($\nu/\kappa$).

![Implied Volatility in the small-epsilon limit](image)
A Cool Application to Forward-Looking Betas

Discrete time CAPM model:

\[ R_a - R_f = \beta_a (R_M - R_f) + \epsilon_a \]

Christoffersen, Jacobs, and Vainberg (2008, McGill University):

\[ \beta_a = \left( \frac{\text{SKEW}_a}{\text{SKEW}_M} \right)^{\frac{1}{3}} \left( \frac{\text{VAR}_a}{\text{VAR}_M} \right)^{\frac{1}{2}} \]

where \( \text{VAR} \) and \( \text{SKEW} \) are variance and risk-neutral skewness
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With Eli Kollman (PhD 2009, UCSB), we propose in

“Calibration of Stock Betas from Skews of Implied Volatilities”

(Applied Mathematical Finance, 2010):

\[ \hat{\beta}_a = \left( \frac{V_{3,a,\epsilon}}{V_{3,M,\epsilon}} \right)^{1/3} = \left( \frac{a_{a,\epsilon}}{a_{M,\epsilon}} \right)^{1/3} \left( \frac{b^{a_*}}{b^{M_*}} \right) \]
**LMMR fits (2/18/2009): S&P500 and Amgen, beta estimate is 1.03**

![Graphs showing the relationship between LMMR and implied volatility for S&P 500 and AMGN.](image)
LMMR fits (2/19/2009): S&P500 and Goldman Sachs, beta estimate is 2.28
THANKS FOR YOUR ATTENTION