Default Clustering and Valuation of CDOs

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York University

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Columbia University

Bachelier 2010
Outline

1. CDO and Review of Pricing Models
   - What is a CDO?
   - Review of Existing Pricing Models

2. The Conditional Survival (CS) Model for CDO Pricing
   - Motivation and Default Clustering
   - Conditional Survival (CS) Model

3. Calibration Results
CDO and Review of Pricing Models

- What is a CDO?
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The Conditional Survival (CS) Model for CDO Pricing

- Motivation and Default Clustering
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Calibration Results
What is a CDO?

- Banks suffered tens of billion dollar losses due to subprime CDOs at the end of 2007
- What is a CDO?
What is a CDO?

Collateralized Debt Obligation (CDO)

- A CDO is a debt security that is constructed from a portfolio of collateral (assets).
What is a CDO?

Morgan Stanley
What is a CDO?

Asset 1
Asset 2

Morgan Stanley

Asset 100
What is a CDO?

Morgan Stanley

Asset 1
Asset 2

Asset 100

CDO Tranches

Tranche 6
22%-100% loss

Tranche 5
12%-22% loss

Tranche 4
9%-12% loss

Tranche 3
6%-9% loss

Tranche 2
3%-6% loss

Tranche 1
0%-3% loss

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Objective of a CDO Pricing Model

- $\tau_i$: default time of the $i$-th name, $i = 1, \ldots, n$
- Cumulative portfolio loss at time $t$:

$$L_t = \sum_{i=1}^{n} c_i \cdot 1_{\{\tau_i \leq t\}}$$

- CDO tranche valuation reduces to calculation of $E[(L_t - K)^+]$
- Objective of CDO pricing model
  - calibrate to single name marginal default probability
  - calibrate to CDO tranche spreads
Two Approaches for CDO Modeling

- Top-down approach builds models for the portfolio cumulative loss process directly
  - Good fit for standard CDO portfolios
  - CANNOT calibrate to single name marginal default probability

- Bottom-up approach builds models for single name default times
  - Consistent with single name marginal default probability
  - Has more difficulty in calibrating CDO tranche spreads
  - Examples: Static bottom-up models, e.g. copula models, dynamic intensity models
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  - Examples: Static bottom-up models, e.g. copula models, dynamic intensity models
Idea: using copula functions to model default time correlation

Literature: Gaussian copula model (Li, 2000)

What is wrong with Gaussian copula?

- Gaussian copula cannot generate tail dependence
  \[
  \lim_{q \to 0} P(\tau_2 < F_2^{-1}(q) | \tau_1 < F_1^{-1}(q)) = 0
  \]

- Gaussian copula does not work during crisis, when the default correlation is strong.
Gaussian Copula Does Not Work During Crisis

Implied Copula Correlation of iTraxx 5Y CDO on 03/14/08

- Tranche: 0%−3%, 3%−6%, 6%−9%, 9%−12%, 12%−22%, 22%−100%
- Implied copula correlation

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Dynamic Intensity Models

- **General idea**
  - Single name default intensity $\lambda_i(t)$: (Jarrow & Turnbull, 1995)
    
    $$P(\tau_i \leq t + \Delta t | \mathcal{F}_t) = \lambda_i(t) \Delta t + o(\Delta t), \text{on } \{\tau_i > t\}$$

  - Building correlation among default intensities $\lambda_1(t), \ldots, \lambda_n(t)$

- **Dynamic intensity model for CDO pricing** (Duffie & Gârleanu, 2001, Mortensen, 2006)
  - $\lambda_i(t) = a_i \lambda^M(t) + \lambda^{id}_i(t), \ i = 1, \ldots, n$
  - $\lambda^M(t)$ and $\lambda^{id}_i(t)$ are independent affine jump diffusion processes

- **Drawback**: cannot incorporate strong default correlation

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Motivation

Default clustering effect: cross-sectional and dynamic (across time)

- The recent demise of major financial institutions
- The iTraxx 5Y index tranche spreads

<table>
<thead>
<tr>
<th>Tranches</th>
<th>0-3%</th>
<th>3-6%</th>
<th>6-9%</th>
<th>9-12%</th>
<th>12-22%</th>
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<tr>
<td>09/20/07</td>
<td>1812</td>
<td>84</td>
<td>37</td>
<td>23</td>
<td>15</td>
<td>7</td>
</tr>
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Dynamic intensity model (Doubly stochastic model):

\[ \tau_i = \inf \{ t \geq 0 : \Lambda_i(t) \geq E_i \}, \quad E_i \sim \exp(1), \text{i.i.d.} \]

\[ \Lambda_i(t) = \int_0^t \lambda_i(s) ds \quad (\Lambda_i(t) \text{ is continuous!}) \]

- **Drawback**: It cannot generate simultaneous defaults of several names.
- Empirical evidence of default clustering exceeding that implied by the doubly stochastic model (Das, Duffie, Kapadia, & Saita, 2007)
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**Empirical observation of simultaneous default:** 24 railway firms defaulted on June 21, 1970 (Azizpour & Giesecke, 2008)

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- \( M(t) = (M_1(t), \ldots, M_J(t)) \): market factor processes that may have jumps
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Conditional survival probability

\[ q^c_i(t) := P(\tau_i > t \mid M(t)) = E \left[ e^{-X^i_{id}(t)} \right] e^{-\sum_{j=1}^{J} a_{i,j}M_j(t)} \]

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Pólya process $M(t)$

- A Pólya process is a mixed Poisson process
- **Clustering jumps**: a Pólya process has positive correlation between increments

\[
\text{Cov}(M(t), M(t + h) - M(t)) > 0
\]

- Discrete integral of CIR process: 
  \[
  M(t) = \int_0^t V(s) \, ds
  \]

\[
dV(t) = \kappa(\theta - V(t)) \, dt + \sigma \sqrt{V(t)} \, dW(t)
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- Laplace transforms of both processes have closed form.
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Simulation is fast: only need to simulate market factor processes and Bernoulli r.v.s

Key fact: conditional on $M(t)$, default events $I_i = 1_{\{\tau_i \leq t\}}$, $i = 1, \ldots, n$ are independent Bernoulli$(1 - q_i^c(t))$ r.v.

Exact simulation of $L_t$ at given time $t$:
1. Generate market factors $M_1(t), M_2(t), \ldots, M_J(t)$.
2. Calculate the conditional survival probability analytically
   \[ q_i^c(t) = q_i(t) \cdot \frac{e^{-\sum_{j=1}^J a_{i,j} M_j(t)}}{E\left[e^{-\sum_{j=1}^J a_{i,j} M_j(t)}\right]} \]
3. Generate independent $I_i \sim$ Bernoulli$(1 - q_i^c(t))$, $i = 1, \ldots, n$.
4. Calculate $L_t = \sum_{i=1}^n c_i \cdot I_i$.

$E[(L_t - K)^+]$: leads to CDO tranche spreads

Control variants: $L_t$
CDO Pricing by Exact Simulation

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- $E[(L_t - K)^+]$: leads to CDO tranche spreads
- Control variants: $L_t$
Sensitivity of CDO tranches w.r.t Single Name CDS

- Sensitivity w.r.t. to single name survival probability

\[
E[(L_t - K)^+] = E[A_i(t)q_i(t) + B_i(t)]
\]

\[
\frac{\partial E[(L_t - K)^+]}{\partial q_i(t)} = E[A_i(t)]
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- \(A_i(t)\) can be obtained as a byproduct in each simulation of \(L_t\).
- The sensitivities w.r.t. each of the \(n\) single name CDS are obtained concurrently with CDO tranche pricing.
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Calibration Results
CDO and CDS data on March 14, 2008, right before the collapse of Bear Stern

Calibration results by using 2 Polya process and 1 discrete integral of CIR process

<table>
<thead>
<tr>
<th>Tranche(%)</th>
<th>0-3</th>
<th>3-6</th>
<th>6-9</th>
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<td>12</td>
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</tr>
</tbody>
</table>

Pricing error: Chi-square = 6.48 (p-value = 0.26), RMSE = 1.11

\[
\text{CHISQ} = \sum_{k=1}^{6} \frac{(s_k - s_k^o)^2}{s_k}, \quad \text{RMSE} = \sqrt{\frac{1}{6} \sum_{k=1}^{6} \left( \frac{s_k - s_k^o}{s_k^{o,a} - s_k^{o,b}} \right)^2}
\]
Model and Market Implied Correlation on 03/14/08

The diagram illustrates the implied copula correlation across different tranche levels on March 14, 2008. The x-axis represents the tranche levels: 0%−3%, 3%−6%, 6%−9%, 9%−12%, 12%−22%, 22%−100%. The y-axis shows the implied copula correlation ranging from 0 to 1. The blue line represents the market data, while the red line represents the CS model. The values at each tranche level indicate the correlation between the default events of the tranches.
CDO and CDS data on September 16, 2008, right after Lehman Brothers went bankruptcy

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<td>Bid-ask spread</td>
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Pricing error: Chi-square = 9.25(p-value = 0.10), RMSE = 2.55
Model and Market Implied Correlation on 09/16/08

![Graph showing implied copula correlation across different tranche segments]

- **Market data**
- **CS model**

Legend:
- Market data
- CS model

Tranche segments:
- 0%−3%
- 3%−6%
- 6%−9%
- 9%−12%
- 12%−22%
- 22%−100%

Y-axis: Implied copula correlation
X-axis: Tranche

Source:
- Peng & Kou (Fields Inst. & Columbia Univ.) Default Clustering and Valuation of CDO

Bachelier 2010
We propose the conditional survival (CS) model:

- It is based on cumulative default intensities instead of intensities.
- It is able to generate a substantially high degree of default clustering.
- It does not specify any dynamics for idiosyncratic default risk component.
- It automatically calibrates to single name marginal default probability.
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Thank you!
### Parameter Stability

**Table:** Compare parameters for 03/14/08 and 09/16/08

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<tr>
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<th>03/14/08</th>
<th>09/16/08</th>
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<tbody>
<tr>
<td>$\alpha_1$</td>
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<td>1.6837</td>
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<tr>
<td>$\lambda(0)$</td>
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<td>1.9176</td>
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</table>
Implicit Constraints on Model Parameters

- The idiosyncratic cumulative intensities $X^{id}_i(t) \geq 0$ and increasing

$$E \left[ e^{-X^{id}_i(T_m)} \right] \leq E \left[ e^{-X^{id}_i(T_{m-1})} \right] \leq \cdots \leq E \left[ e^{-X^{id}_i(T_1)} \right] \leq 1, \forall 1 \leq i \leq n$$

- Recall that

$$E \left[ e^{-X^{id}_i(t)} \right] = \frac{q_i(t)}{E \left[ e^{-\sum_{j=1}^{J} a_{i,j} M_j(t)} \right]}$$

- This imposes parameter constraints:

$$\frac{q_i(T_m)}{E \left[ e^{-\sum_{j=1}^{J} a_{i,j} M_j(T_m)} \right]} \leq \cdots \leq \frac{q_i(T_1)}{E \left[ e^{-\sum_{j=1}^{J} a_{i,j} M_j(T_1)} \right]} \leq 1, \forall 1 \leq i \leq n$$
Calibration Algorithm

Pricing error function: $F(\Theta)$

1. Initialization: set market factor parameter $\Theta_0$, and set $s = 0$.
2. Iteration: $s \rightarrow s + 1$
   - For given $\Theta_s$, determine loading coefficients $a_{i,j}$ by solving a constrained optimization problem:

   $\min \ E \left[ e^{-\sum_{j=1}^{J} a_{i,j}M_j(T_m)} \right] - q_i(T_m)$
   
   $s.t. \quad \frac{q_i(T_m)}{E \left[ e^{-\sum_{j=1}^{J} a_{i,j}M_j(T_m)} \right]} \leq \cdots \leq \frac{q_i(T_1)}{E \left[ e^{-\sum_{j=1}^{J} a_{i,j}M_j(T_1)} \right]} \leq 1$
   
   $0 \leq a_{i,j}$

   - Calculate the tranche spreads and pricing error $F(\Theta_s)$
   - Update the market factor parameter $\Theta_s \rightarrow \Theta_{s+1}$ by some optimization routine, e.g. Powell’s direction-set algorithm
   - Repeat