Exact Sampling of Jump-Diffusion Processes

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Jump-Diffusion Processes

- Ubiquitous in finance and economics
  - Price models: equity, commodity, rates, energy, FX
  - Default timing models (jump component)

- We develop a method for the exact sampling of a jump-diffusion process with state-dependent drift, volatility, jump intensity, and jump size
  - Leads to unbiased simulation estimators of derivative prices, risk measures, and other quantities

- The method extends an innovative acceptance/rejection scheme developed by Beskos & Roberts (2005, AAP) and Chen (2009) for diffusions
Jump-Diffusion

- Fix a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\)
- For suitable functions \(\mu\) and \(\sigma\), consider the SDE

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t
\]

where \(W\) is a standard Brownian motion and \(J\) is a jump process

\[
J_t = \sum_{n \leq N_t} c(X_{T_n^-}, Z_n)
\]

- \(N\) is a counting process with event times \((T_n)\) and intensity 
  \(\lambda_t = \Lambda(X_{t^-})\) for a suitable function \(\Lambda\)
- \((Z_n)\) is a sequence of mark variables valued in \(E\)
- \(c : D_X \times E \rightarrow D_X\) determines the jump magnitudes

- \(J\) is self-exciting; dependence between jump sizes and frequency

Kay Giesecke
Jump-Diffusion

- For a suitable function $f$ and a horizon $T$ we wish to calculate

$$\mathbb{E}\{f(X_T, (J_t)_{t \leq T})\}$$

- Price of a derivative written on $X_T$
- Price of a credit derivative written on $J$

- Alternative approaches
  - Analytical solutions: rare; e.g. Merton and Kou models
  - Semi-analytical transform approaches: AJDs, LQJDs, Lévy
  - PIDE approaches: fewer restrictions
  - Monte Carlo simulation: perhaps the widest scope
Discretizing the Jump-Diffusion

Standard approach to simulating $X$

- $X$ is approximated on a discrete-time grid
  - Euler or higher order scheme for diffusion component
  - Thinning or time-scaling scheme for jump times $T_n$

- Simulation estimator is biased
  - Magnitude of the bias? Confidence intervals?
  - Convergence of scheme?
  - Allocation of computational budget?
Discretizing the Jump-Diffusion

Time-scaling for jumps: $T_n \overset{d}{=} \inf\{t : \int_0^t \Lambda(X_s)ds \geq \mathcal{E}_1 + \cdots + \mathcal{E}_n\}$
Exact Sampling

• We provide an exact sampling scheme for $X$ that avoids discretization entirely, and leads to unbiased simulation estimators

• First step: transform $X$ into a unit-diffusion SDE
  - Lamperti transform $F(x) = \int_{X_0}^{x} \frac{1}{\sigma(u)} du$
  - Then $Y_t = F(X_t)$ solves
    \[ dY_t = \alpha(Y_t) dt + dW_t + dJ^Y_t \]

where

\[ \alpha(y) = \frac{\mu(F^{-1}(y))}{\sigma(F^{-1}(y))} - \frac{1}{2} \sigma'(F^{-1}(y)) \]

\[ J^Y_t = \sum_{n \leq N_t} \Delta(Y_{T_n}, Z_n) \]

for $\Delta(y, z) = F(F^{-1}(y) + c(F^{-1}(y), z)) - y$
Exact Sampling

Assumptions

1. $\sigma(x)$ is bounded away from 0

2. $\mu(x)$ and $\Lambda(x)$ are continuously differentiable and $\sigma(x)$ is twice continuously differentiable.

3. Conditions on $\alpha(y)$ and $c(x, z)$ guaranteeing that $Y$ does not reach the boundaries in finite time (known).
Acceptance/Rejection Scheme

- The exact method uses an A/R scheme
- Suppose we want to sample from a density $f(y)$ and there is another density $g(y)$ and a constant $c > 0$ such that

$$c \cdot \frac{f(y)}{g(y)} \leq 1$$

- A/R scheme
  1. Draw a sample $Y$ from $g$
  2. Draw a Bernoulli variable $I$ with success probability $c \cdot \frac{f(Y)}{g(Y)}$
  3. Accept $Y$ as a sample from $f$ if $I = 1$
A/R for Continuous $Y$

Beskos & Roberts (2005, AAP)

- We wish to sample $Y_T$ using the A/R scheme
- Under Novikov and additional boundedness conditions,

$$\frac{f_{Y_T}(y)}{g(y)} \propto \mathbb{E} \left[ \exp \left( - \int_0^T \phi(W_s) ds \right) \middle| W_T = y \right] =: H(y)$$

where $\phi = (\alpha' + \alpha^2)/2$ and $g(y)$ is a proposal density
- For the A/R step, note that $H(y) = \mathbb{P}(M_T = 0 \mid W_T = y)$ where $M$ is a doubly-stochastic Poisson process with intensity $\phi(W_s)$
  - The Bernoulli indicator $\{I = 1\} = \{M_T = 0\}$ can be generated by sampling $M_T$ given $W_T = Y$ with $Y$ drawn from $g$
  - If $\phi(x) \leq \pi$, then this can be done by thinning a Poisson process with intensity $\pi$ (requires sampling from BB)
A/R for Continuous $Y$

Localization: Chen (2009)
A/R for Continuous \( Y \)

Generating the first piece of \( Y \)

- Target exit time \( \zeta_1 = \inf\{t \geq 0 : |Y_t - Y_0| \geq L\} \) for \( L > 0 \)
- Proposal exit time \( \tau = \inf\{t \geq 0 : |W_t| \geq L\} \)
- The LR between \((\zeta_1, Y_1 - Y_0)\) and \((\tau, W_\tau)\) is proportional to

\[
\mathbb{E}\left[ \exp\left( - \int_0^\tau \phi(Y_0 + W_s)ds \right) \mid \tau, W_\tau \right]
\]

- Because of the continuity assumptions, \( \phi(Y_0 + W_s) \) is bounded and thinning can always be used in the acceptance test
  - Need to sample from Brownian meander
- The density of \( \tau \) is known and can be sampled from using the method of Burq & Jones (2006)
A/R for Jump-Diffusion $Y$

Introducing jumps

![Diagram](image-url)
A/R for Jump-Diffusion $Y$

Generating the first piece of $Y$

1. Sample $\tau$ as before using a bound $L$. Suppose $\tau \leq T$.
2. Sample candidate jump times $(\sigma_1, \ldots, \sigma_n)$ of $Y$ during $[0, \tau]$ from a Poisson process with intensity $\pi \geq \Lambda(F^{-1}(Y_0 + y))$, $y \in [-L, L]$.
3. Sample $(W_{\sigma_1}, \ldots, W_{\sigma_n}, W_\tau)$ from a Brownian meander.
4. Perform acceptance tests for the $\sigma_i$ by drawing Bernoulli variables with success probabilities $\Lambda(F^{-1}(Y_0 + W_{\sigma_i}))/\pi$.
5. Perform acceptance test for $(\zeta_1, Y_{\sigma_1} - Y_0, \ldots, Y_{\sigma_k} - Y_0)$ given the proposal $(\tau, W_{\sigma_1}, \ldots, W_{\sigma_k})$, where $\sigma_k$ is the first candidate jump time of $Y$ accepted in the previous step.
6. If the skeleton is accepted, draw mark $Z_1$ and compute $Y_{T_1} = Y_{\sigma_k} + \Delta(Y_{\sigma_k}, Z_1)$.
A/R for Jump-Diffusion $Y$

Likelihood ratio

- The LR for the last acceptance test is proportional to

$$e^{A(Y_0+W_{\sigma_k})} \mathbb{E} \left[ \exp \left( - \int_0^{\sigma_k} \phi(Y_0 + W_u) du \right) \middle| \tau, W_{\sigma_1}, \ldots, W_{\sigma_k} \right]$$

where $A(x) = \int_0^x \alpha(u) du$ and $\phi = (\alpha' + \alpha^2)/2$

- Generate Bernoulli indicator by generating the jump times of a doubly-stochastic Poisson process with intensity $\phi(Y_0 + W_u)$
  - Thinning applies
  - Sample from Brownian meander
Numerical examples


\[ dX_t = (r + \Lambda(X_t))X_t dt + \sigma(X_t) dW_t + dJ_t \]

where \(X_0 > 0\) and for \(a > 0, b \geq 0, c \geq \frac{1}{2}\) and \(\beta < 0\)

- \(\Lambda(x) = b + ca^2 x^{2\beta}\) is the jump intensity
- \(\sigma(x) = ax^{\beta+1}\) is the volatility
- \(c(x, z) = -xz\) for \(z \in (0, 1)\) is the jump size

- The firm defaults at the first jump time \(T_1\) of \(J\)

- The default intensity \(\lambda = \Lambda(X)\) is unbounded

  - Violates boundedness hypothesis of thinning scheme for jumps
    (Glasserman & Merener (2003), Casella & Roberts (2010))
  - Convergence order of discretization scheme unknown
Numerical examples

- We are interested in $X$ during $[0, T \wedge T_1]$ for some $T > 0$
- The target functional takes the form

$$f(X_T, (J_t)_{t \leq T}) = h_1(X_T)1_{\{J_T=0\}} + h_2(X_T)1_{\{J_T \neq 0\}}$$

Examples

- Probability of survival to $T$: $h_1(x) = 1$ and $h_2(x) = 0$
- European put with strike $K$ and maturity $T$:
  $$h_1(x) = e^{-rT}(K - x)^+ \quad \text{and} \quad h_2(x) = Ke^{-rT}$$

- Carr & Linetsky (2006) provide analytical solutions to these and other quantities
Numerical examples

- We estimate the price of a European put on $X$
- We consider the RMSE $= \sqrt{\text{Bias}^2 + \text{SE}^2}$ for
  - The exact method, for which the bias is 0
  - The discretization method (Euler plus time-scaling for jumps)
    * The number of time steps is equal to the square root of the number of trials (Duffie & Glynn (1995))
    * The bias is estimated using 10 million trials
- Matlab implementation (favors discretization)
Numerical examples

\[ X_0 = 50, \beta = -1, r = 0.05, a = 50/4, b = 0, c = 0.5, T = 1, \text{ strike } 5, \text{ analytical value } 0.1491 \]

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<th>Steps</th>
<th>Value</th>
<th>Bias</th>
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Numerical Examples

Convergence of RMSEs (log-log plot), strike $K = 5$
Numerical examples

\[ X_0 = 50, \beta = -1, r = 0.05, a = 50/4, b = 0, c = 0.5, T = 1, \text{ strike 50, analytical value } 4.4118 \]

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Numerical Examples

Convergence of RMSEs (log-log plot), strike $K = 50$
Extensions

• The target functional can take the form

\[ \mathbb{E}\{f((X_t)_{t \in S}, (J_t)_{t \leq T})\} \]

for a discrete set \( S \) of times \( t \in [0, T] \)

– Treatment of certain path-dependent payoffs

• The intensity can take the form

\[ \lambda_t = \Lambda(X_{t-}, J_{t-}, t) \]
Conclusions

• We develop a method for the exact sampling of a one-dimensional jump-diffusion process with state-dependent drift, volatility, jump intensity and jump size
  – Only mild conditions on the coefficients are required

• Numerical experiments indicate the advantages of the method over a conventional discretization scheme

• Future research
  – Efficiency: choice of localization bound
  – Extension to multiple dimensions: stochastic volatility with state-dependent jumps