Pricing and Hedging with Constant Elasticity and Stochastic Volatility

Sun-Yong Choi\textsuperscript{1} Jean-Pierre Fouque\textsuperscript{2} Jeong-Hoon Kim\textsuperscript{3}

\textsuperscript{1,3}Department of Mathematics 
Yonsei University

\textsuperscript{2}Department of Statistics and Applied Probability 
University of California, Santa Barbara

6th World Congress of the Bachelier Finance Society 
Toronto, Canada
Introduction

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Drawback of Black Scholes Model

- Smile curve

- Rubinstein (1985)
- Jackwerth and Rubinstein (1996)
- In B.S. Model, Implied Volatility curve is flat.
- We need to use the implied volatility which explicitly depends on the option strike and maturity.
Local Volatility Model

One needs volatility to depend on underlying
Local Volatility Models

\[
\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dW_t
\]

The dynamics of Implied Volatility in Local Volatility model.

This is opposite to real market. (Hagan, 2002)
CEV Model

- CEV (constant elasticity of variance) diffusion model
- $X_t$ stock price s.t.

$$dX_t = \mu X_t dt + \sigma X_t^{\frac{\theta}{2}} dW_t$$

- Introduced by Cox and Ross (1976)

- Analytic tractability
CEV Model cont.

- When $\theta = 2$ the model is Black-Scholes case.
- When $\theta < 2$ volatility falls as stock price rises. ⇒ realistic, can generate a fatter left tail.
- When $\theta > 2$ volatility rise as stock price rises. ⇒ (futures option)
CEV Model cont.

**Theorem (Lipton, 2001)**

The call option price $C_{CEV}$ for $X_t = x$ is given by

$$C_{CEV}(t, x) = e^{-r(T-t)} x \int_\tilde{K}^\infty \left( \frac{\tilde{x}}{y} \right)^{\frac{1}{2(2-\theta)}} e^{-(\tilde{x}+y)I_{\frac{1}{2-\theta}}(2\sqrt{\tilde{x}y})} dy$$

$$+ e^{-r(T-t)} K \int_\tilde{K}^\infty \left( \frac{y}{\tilde{x}} \right)^{\frac{1}{2(2-\theta)}} e^{-(\tilde{x}+y)I_{\frac{1}{2-\theta}}(2\sqrt{\tilde{x}y})} dy,$$

where

$$\tilde{x} = \frac{2xe^{r(2-\theta)(T-t)}}{(2-\theta)^2 \chi}, \quad \chi = \frac{\sigma^2}{(2-\theta)r} \left( e^{r(2-\theta)T} - e^{r(2-\theta)t} \right)$$

$$\tilde{K} = \frac{2K^{2-\theta}}{(2-\theta)^2 \chi}$$
Dynamics of Implied Volatility for CEV model

\[ \theta = 1.9 \]
\[ \theta = 2.1 \]
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Stochastic Volatility CEV model

Underlying asset price Model $X_t$ and $Y_t$

\[ dX_t = \mu X_t dt + f(Y_t)X_t^{\theta} dW_t \]  

\[ dY_t = \alpha(m - Y_t) dt + \beta d\hat{Z}_t, \]  

where $f(y)$ smooth function and the Brownian motion $\hat{Z}_t$ is correlated with $W_t$ such that

\[ d<W, \hat{Z}>_t = \rho dt. \]  

In terms of the instantaneous correlation coefficient $\rho$, we write

\[ \hat{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t \]
Characteristics

- The new volatility is given by the multiplication of a function of a new process.
- The new process is taken to be an Ito process (O-U process):
  \[ dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t. \]
- \( \alpha = \) rate of mean reversion.
  Assume that mean reversion is fast.
  So, \( \alpha \) is large enough.
Use Risk Neutral Valuation method

Equivalent martingale measure $Q$, Option price is given by the formula

$$P(t, x, y) = E^Q[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y]$$ (5)
Theorem 3.1

The option price $P(t, x, y)$ defined by (5) satisfies the Kolmogorov PDE

\[
P_t + \frac{1}{2} f^2(y) x^\theta P_{xx} + \rho \beta f(y) x^{\frac{\theta}{2}} P_{xy} + \frac{1}{2} \beta^2 P_{yy} \\
+ r x P_x + (\alpha (m - y) - \beta \Lambda(t, x, y)) P_y - rP = 0, \quad (6)
\]

where

\[
\Lambda(t, x, y) = \rho \frac{\mu - r}{f(y) x^{\frac{\theta-2}{2}}} + \sqrt{1 - \rho^2 \gamma(y)}.
\]
Asymptotic theory

- Develop an asymptotic theory on fast mean reversion
- Introduce a small parameter $\epsilon$

$$\epsilon = \frac{1}{\alpha}$$

- Assume $\nu = \frac{\beta}{\sqrt{2\alpha}}$ is fixed in scale as $\epsilon$ become zero.

$$\alpha \sim O(\epsilon^{-1}), \quad \beta \sim O(\epsilon^{-1/2}), \quad \text{and} \quad \nu \sim O(1).$$
To solve the PDE (6), we use **Singular Perturbation** method.

### Procedure

- Substituting the asymptotic series

\[ P(x; \epsilon) \approx \sum_{n=0}^{\infty} \epsilon^n P_n(x) \]

into the differential equation.

- Expanding all quantities in a power series in \( \epsilon \).

- Collecting terms with same powers of \( \epsilon \) and equating them to zero.

- Solving this hierarchy of the problem sequentially.
Asymptotic theory

After rewritten in terms of $\epsilon$, the PDE (6) becomes

$$
Pt + \frac{1}{2} f^2(y)x^\theta P_{xx} + \rho \frac{\sqrt{2} \nu}{\sqrt{\epsilon}} f(y)x^\theta P_{xy} + \frac{1}{2} \frac{2 \nu^2}{\epsilon} P_{yy} \\
+ rxP_x + \left( \frac{1}{\epsilon} (m - y) - \beta \frac{\sqrt{2} \nu}{\sqrt{\epsilon}} \Lambda(t, x, y) \right) P_y - rP = 0
$$

Collecting by $\epsilon$ order,

$$
\frac{1}{\epsilon} \left( \nu^2 P_{yy} + (m - y) P_y \right) + \frac{1}{\sqrt{\epsilon}} \left( \sqrt{2} \rho \nu f(y)x^\theta P_{xy} - \sqrt{2} \nu \Lambda P_y \right) \\
+ \left( P_t + \frac{1}{2} f^2(y)x^\theta P_{xx} + rxP_x - rP \right) = 0
$$
Define operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ as

\begin{align*}
\mathcal{L}_0 &= \nu^2 \partial_{yy}^2 + (m - y) \partial_y, \\
\mathcal{L}_1 &= \sqrt{2} \rho \nu f(y) x^\theta \partial_{xy}^2 - \sqrt{2} \nu \Lambda(t, x, y) \partial_y, \\
\mathcal{L}_2 &= \partial_t + \frac{1}{2} f(y)^2 x^\theta \partial_{xx}^2 + r(x \partial_x - \cdot).
\end{align*}

then, the PDE (6) can be written as

\[ \left( \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P^\epsilon = 0 \]
Asymptotic Expansion

Expand $P^\epsilon$ in powers of $\sqrt{\epsilon}$:

$$P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \cdots$$  \hspace{1cm} (11)

Here, the choice of the power unit $\sqrt{\epsilon}$ in the power series expansion was determined by the method of matching coefficient.
Substituting the PDE (10),

\[
\frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) \\
+ (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \\
\sqrt{\epsilon} (\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots = 0,
\]

which holds for arbitrary $\epsilon > 0$. 
Lemma 3.1

If solution to the Poisson equation

\[ \mathcal{L}_0 \chi(y) + \psi(y) = 0 \]  

exists, then the following centering (solvability) condition must satisfy \( \langle \psi \rangle = 0 \), where \( \langle \cdot \rangle \) is the expectation with respect to the invariant distribution of \( Y_t \). If then, solutions of (13) are given by the form

\[ \chi(y) = \int_0^t E^y[\psi(Y_t)] \, dt + \text{constant}. \]  

Note:

\[ \langle \psi \rangle = \int_{-\infty}^{\infty} \psi(y) f(y) \, dy, \quad f(y) = \frac{1}{\sqrt{2\pi\nu^2}} \exp \left( -\frac{(y - m)^2}{2\nu^2} \right) \]
From the asymptotic expansion (12) $1/\epsilon$ order, we first have

$$\mathcal{L}_0 P_0 = 0. \quad (15)$$

Solving this equation yields

$$P_0(t, x, y) = c_1(t, x) \int_0^y e^{\frac{(m-z)^2}{2\nu^2}} dz + c_2(t, x)$$

for some functions $c_1$ and $c_2$ independent of $y$.

- $c_1 = 0$ is required.
- $P_0(t, x, y)$ must be a function of only $t$ and $x$

$$P_0 = P_0(t, x).$$
Asymptotic Expansion cont.

- From the expansion (12) $1/\sqrt{\epsilon}$ order,

$$\mathcal{L}_0P_1 + \mathcal{L}_1P_0 = 0$$

- Known $\mathcal{L}_1P_0 = 0$
- Get $\mathcal{L}_0P_1 = 0$

$$P_1 = P_1(t, x) \quad (16)$$
The leading term $P_0(t, x)$ is given by the solution of the PDE

$$\frac{\partial P_1}{\partial t} + \frac{1}{2} < f^2 > x^\theta \frac{\partial^2 P_1}{\partial x^2} + r(x \frac{\partial P_1}{\partial x} - P_1) = 0 \quad (17)$$

with the terminal condition $P_0(T, x) = h(x)$. 

Proof of Theorem 3.2

Proof
From the expansion (12), the PDE

\[ \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0 \]  \hspace{1cm} (18)

Since \( \mathcal{L}_1 P_1 = 0 \), then

\[ \mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0 \]  \hspace{1cm} (19)

which is a Poisson equation.
Proof of Theorem 3.2 cont.

From Lemma 3.1 with $\psi = \mathcal{L}_2 P_0$, $P_0(t, x)$ has to satisfy the centering condition

$$<\mathcal{L}_2> P_0 = 0$$

with the terminal condition $P_0(T, x) = h(x)$, where

$$<\mathcal{L}_2> = \partial_t + \frac{1}{2} <f^2> x^\theta \partial_{xx}^2 + r(x\partial_x - \cdot).$$

Thus $P_0$ solves the PDE (17). $\square$
Asymptotic Expansion $P_1$ cont.

Theorem 3.3

The first correction $P_1(t, x)$ is given by the solution of the PDE

$$\frac{\partial P_1}{\partial t} + \frac{1}{2} \langle f^2 \rangle x^\theta \frac{\partial^2 P_1}{\partial x^2} + r(x \frac{\partial P_1}{\partial x} - P_1) =$$

$$V_3x \frac{\partial}{\partial x}(x^2 \frac{\partial^2 P_0}{\partial x^2}) + V_2x^2 \frac{\partial^2 P_0}{\partial x^2}$$

(21)

with the final condition $P_1(T, x) = 0$, where $V_3$ and $V_2$ are given by (22) and (23), respectively.
Asymptotic Expansion $P_1$

For convenience,

$$V_3(x; \theta) = \frac{\nu}{\sqrt{2}} \rho x^{\frac{\theta-2}{2}} <f \psi_y>, \quad (22)$$

$$V_2(x; \Lambda; \theta) = \frac{\nu}{\sqrt{2}} \left( \rho x^{\frac{\theta}{2}} <f \psi_{xy}> - <\Lambda \psi_y> \right), \quad (23)$$

where $\psi(t, x, y)$ is solution of the Poisson equation

$$\mathcal{L}_0 \psi = \nu^2 \psi_{yy} + (m - y) \psi_y = (f^2 - <f^2>) x^{\theta-2}. \quad (24)$$
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\( \theta = 1.95 \text{ and } \epsilon = 0.01 \)

Line 1 = \( P_0 \), Line 2 = \( P_1 \), Line 3 = \( P_0 + \sqrt{\epsilon}P_1 \)
\( \theta = 2.00 \) and \( \epsilon = 0.01 \)

\[ \theta_1 = 2.0, \epsilon = 0.01 \]

\( P_0(t, x) \) with \( K = 100, \ T = 1, \ \overline{\sigma} = 0.165, \) and \( r = 0.02. \) It is computed with terminal condition \( h(x) = (x - K)^+ \).
$\theta = 2.05$ and $\epsilon = 0.01$
\( \theta = 1.95 \), \( \theta = 1.95 \), \( \theta = 2.05 \)

Line 1: \( \theta = 1.95 \), Line 2: \( \theta = 1.95 \), Line 3: \( \theta = 2.05 \)
Dynamics of Implied Volatility ($\theta = 1.9$ and $\theta = 1.925$)

Line 1: $X_0 = 90$,  Line 2: $X_0 = 95$,  Line 3: $X_0 = 100$,  
Line 4: $X_0 = 105$,  Line 5: $X_0 = 110$
Dynamics of Implied Volatility ($\theta = 1.95$ and $\theta = 1.975$)
Dynamics of Implied Volatility ($\theta = 2.00$ and $\theta = 2.025$)
Dynamics of Implied Volatility ($\theta = 2.05$ and $\theta = 2.075$)
Remark

- Implied volatility curve move from left to right for $\theta \geq 1.975$.
- For $\theta \geq 2$, The implied volatility curve seems to be skew, unlike CEV model.
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- Corrected Price (A new hybrid model)
- Right dynamics of Implied Volatility
- Stability of Hedging
- Still ongoing research (Fitting to Market Data)
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Patrick S. Hagan, Deep Kumar. Andrew S. Lesniewski, Diana E. Woodward : Managing Smile Risk
Thank you for your attention!