An Equilibrium Model for Term Structure of Interest Rates with Consumption Commitments

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“Free market long-term rates of interest for any industrial nation, properly charted, provide a sort of fever chart of the economic and political health of that nation.”
Yield of a bond

is the annual continuous compound return rate. For a (zero coupon) bond with term $i$, current price $p$ and pays one unit at maturity, denote the yield by $y(i)$. We have $pe^{y(i)i} = 1$ or

$$y(i) = \frac{-\ln(p)}{i}.$$
Term structure refers to the function relationship $i \rightarrow y(i)$. It is often graphically represented by $\text{graph}(y)$ – a yield curve.
Existing term structure models

- Short rate model.
- No arbitrage model.
The short rate model

- Assuming a model for the short period bonds in to the future, derive term structures according to principle of consistency–no arbitrage opportunities.

- O. Vasicek pioneered this model in 1977.


- Simplicity is a tradition in this line of research.
No arbitrage model

- Using the current yield curve as a starting point, derive term structures according to principle of no arbitrage.
- Ho and Lee (1986) and Hull and White (1990) are representatives of early works in this direction.
Our approach: Motivation

to reflect the impact of consumption commitment. Examples:

- Saving for education and retirement.
- Insurance companies projected need of funds due to their insurance portfolio.
- Foreign countries demand commensurable with their structure needs.
Since our focus is on the supply and demand in nature, a general equilibrium model becomes a natural choice.

We use a model that develops the influential Lucas (1978) model for stock pricing.

Comparing to the existing models, this model is more general in principle

but, in this initial investigation, less sophisticated technically.
Results

- Establish dynamic programming equations
- Existence and uniqueness of the equilibrium term structures.
- Computation algorithms.
- Sample testing results that are consistent with intuition and qualitative policy ramifications.
Related extension of the Lucas model

Model of Judd, Kubler and Schmedders (2003)

- Also involve both stock and bonds.
- Focuses on Perato inefficiency of the equilibrium, portfolio separation and trading volume.
- Involving multiple agents – generalizes the Lucas model in a different direction.
- Our model use only one representative agent as Lucas and is computationally tractable.
Model: The producer

The producer is the aggregate of all agents producing consumption goods. We represent it as one unit share of stock whose products in each period can be regarded as a dividend process $D_t$. We assume that $D_t$ is a stationary markovian process defined by its transition function

$$F(D', D) = \text{Prob}(D_{t+1} \leq D' \mid D_t = D).$$

(1)

The ex-dividend price at $t$ of the stock is denoted by $s_t$. $D_t$ is the sole source of uncertainty in this model.
Assumption on the dividend process

We make the following technical assumption on $D = D_t$.

**(A1)** The dividend $D_t$ has a compact positive range $[D_m, D_M]$ and that the mapping $T : C([D_m, D_M], R^m) \rightarrow C([D_m, D_M], R^m)$ defined by

$$(Tg)(D) = \int g(D')dF(D', D)$$

maps any bounded set in $C([D_m, D_M], R^m)$ to an equi-continuous subset of $C([D_m, D_M], R^m)$. Condition (A1) is automatically satisfied when $D$ has only a finite number of states.
Model: The government

fund its operation through taxation $qD_t$ and issuing (zero coupon) bond with duration 1 through $m$ periods. The vector

$$p_t(D_t) = (p_t(1, D_t), \ldots, p_t(m, D_t))^\top$$

represents the prices of the $m$ bonds. We will use vector

$$B_t = (B_t(1), \ldots, B_t(m))^\top$$

to denote the quantity of outstanding bonds at $t$. These bonds will be purchased by the consumer and acting as a medium for transferring consumption power.
Assumption on par value of the bonds

The total par value of the government bonds is assumed to be a fixed portion of the total expected production, that is

$$\bar{1} \cdot B_t = \sum_{i=1}^{m} B_t(i) = \alpha E[D_t] = \alpha E[D], \quad (2)$$

where $\alpha$ reflects the government fiscal policy towards debt. We can roughly think $\alpha$ as a proxy for the ratio of total government debt to GDP.
Model: The consumer

wishes to maximize

\[ E \left[ \sum_{t=0}^{\infty} \beta^t (u(c_t) + \phi(a_t - \gamma_t)) \right]. \]  

(3)

c_t \text{ – stochastic processes of consumption.}

a_t \text{ – allocation for (partially) satisfying the commitment.}

\beta \text{ – discount factor.}

u \text{ – concave utility function}

\phi \text{ – concave penalty function for deviation from commitments.}

w_t \text{ – net worth at time } t.

\gamma_t \text{ – the consumption commitment at period } t.
The consumer distributes his consumption power among different time periods by holding and adjusting a portfolio

\[(\theta_t, \psi_t) = (\theta_t, \psi_t(1), \ldots, \psi_t(m))\]

in which \(\theta_t\) represents the share of stocks and \(\psi_t(i), i = 1, 2, \ldots, m\) represents the share of the \(i\) period bond. In general, \(c_t, a_t, \theta_t\) and \(\psi_t\) are all depend on \(D_t\).
Assumptions on the penalty function

The penalty function $\phi(s)$ attains its maximum at $s = 0$ and $\phi(0) = 0$. Insufficient allocation to meet the commitment indicated by $a_t - \gamma_t < 0$ will be penalized. Assuming $\phi$ to be concave means that as the deficit in satisfying the consumption commitment $a_t - \gamma_t$ grows larger, the rate of marginal penalty will increase. With this model there is no incentive to allocate $a_t$ above the consumption commitment $\gamma_t$ because doing so will reduce the utility and increase the penalty $(\text{reduce the value of } \phi(a_t - \gamma_t))$. When $\gamma_t = 0$ this means that we will allow $a_t$ to take negative value.
Model: Constraints

No negative consumption:

\[ c_t \geq 0. \]  \hspace{1cm} (4)

Budget constraint: for \( t = 0, 1, \ldots, \)

\[ c_t(D_t) + a_t(D_t) + \theta_t(D_t)s_t(D_t) + \psi_t(D_t) \cdot p_t(D_t) \leq w_t, \]  \hspace{1cm} (5)

where, \( w_0 \) is given and, for \( t = 1, 2, \ldots, \)

\[ w_t = \theta_{t-1}(D_{t-1})(s_t(D_t) + (1 - q)D_t) + \psi_{t-1}^1(D_{t-1}) \]
\[ + \psi_{t-1}^2(D_{t-1})p_t^1(D_t) + \ldots + \psi_{t-1}^m(D_{t-1})p_t^{m-1}(D_t). \]

The consumer is not allowed to short or issue bonds so that

\[ \psi_t(i) \in [0, B_t(i)] \] for all \( t \) and \( i \).
\[ w_t = \theta_{t-1}(D_{t-1})(s_t(D_t) + (1 - q)D_t) + \psi_{t-1}(D_{t-1}) \cdot (Ap_t(D_t) + b), \]

where \( A \) and \( b \) are defined as follows

\[
A = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
\end{bmatrix}
\text{ and } b = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}.
\]
Model: Equilibrium quantities

In equilibrium: The consumer holds all the shares of stock and bonds at all time and consumes all the after-tax dividend of the stock and interest of the bonds in each period. That is, for all $t$,

$$\theta_t = 1, \psi_t = B_t$$

and

$$c_t(D_t) + a_t(D_t) = (1 - q)D_t + B_{t-1} \cdot (Ap_t(D_t) + b) - B_t \cdot p_t(D_t).$$

Price process $p_t$ (which determines the term structure) induces such a general market equilibrium is the equilibrium we are seeking.
The quantity

\[ g_t = B_t \cdot p_t(D_t) - B_{t-1} \cdot (A p_t(D_t) + b) \]

is the government’s additional expenditure financed by issuing bonds. When \( g_t > 0 \), the government consumes additional \( g_t \) from the consumer’s portion of the dividend and when \( g_t < 0 \) the government yield part of its portion of the dividend to consumers.
Model: Optimization problem

maximize \( E \left[ \sum_{t=0}^{\infty} \beta^t [u(c_t) + \phi(a_t - \gamma_t)] \right] \)  \( (6) \)

subject to

\[
\begin{align*}
c_t &\geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, t = 0, 1, \ldots, \\
c_t(D_t) + a_t(D_t) + \theta_t(D_t)s_t(D_t) + \psi_t(D_t) \cdot p_t(D_t) &\leq \theta_{t-1}(D_{t-1})(s_t(D_t) + (1 - q)D_t) \\
&+ \psi_{t-1}(D_{t-1}) \cdot (Ap_t(D_t) + b), t > 0, \\
c_0(D_0) + a_0(D_0) + \theta_0(D_0)s_0(D_0) + \psi_0(D_0) \cdot p_0(D_0) &\leq w_0.
\end{align*}
\]

We assume that the initial state \( D_0 \) and the initial endowment \( w_0 \) are fixed.
Denote the optimal value function by $v(w_0, D_0, \vec{\gamma})$ where $\vec{\gamma} = (\gamma_0, \gamma_1, \ldots)$. Using $S$ to signify the right shifting operator we have dynamic programming equations, for $t = 0, 1, \ldots$,

\[
v(w, D, S^t \vec{\gamma}) = \max \left[ u(c_t) + \phi(a_t - \gamma_t) + \beta \int v(w', D', S^{t+1} \vec{\gamma}) dF(D', D) \right]
\]

s.t.

\[
c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m,

c_t + a_t + \theta_t s_t(D) + \psi_t \cdot p_t(D) \leq w
\]

\[
\theta_t s_{t+1}(D') + (1 - q)D' + \psi_t \cdot (A p_{t+1}(D') + b) = w'.
\]
Optimal value as functions of the portfolio

Given $\vec{s} = (s_0, s_1, \ldots)$ and $\vec{p} = (p_0, p_1, \ldots)$,

$$V(\theta, \psi, D, \vec{\gamma}|\vec{s}, \vec{p}) = v(\theta(s_0(D) + (1 - q)D) + \psi \cdot (A p_0(D) + b), D, \vec{\gamma}),$$

defines the optimal value as a function of the portfolio. We emphasize that

1. the portfolio $(\theta, \psi)$ is independent on $D_{t+1}$, since it is generated by the optimization problem for the period $t$ and depends only on $D = D_t$, and

2. the price processes $\vec{s}$ and $\vec{p}$ relates the portfolio and the wealth level and, therefore, they must be specified in the definition of $V$. 
for \( t = 0, 1, \ldots \),

\[
V(\theta, \psi, D, S^t \bar{\gamma} | S^t \bar{s}, S^t \bar{p}) = \max \left[ u(c) + \phi(a - \gamma_t) + \beta \int V(\theta', \psi', D', S^{t+1} \bar{\gamma} | S^t \bar{s}, S^t \bar{p}) dF(D', D) \right]
\]

s.t. \( c \geq 0, \psi'(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, \)
\( c + a + \theta' s_t(D) + \psi' \cdot p_t(D) \leq \theta(s_t(D) + (1 - q)D) + \psi \cdot (A p_t(D) + b) \).
Finite number of different consumption commitments

Under reasonable conditions we can solve for the equilibrium for the practically useful case in which there are only finite number of different consumption commitments, say all within $n$, using the following steps:
There are only $n + 1$ different dynamical programming equations now. They are, for $t = 0, 1, \ldots, n - 1$,

\[
\nu(w, D, S_{t}^\gamma) = \\
\max \left[ u(c_t) + \phi(a_t - \gamma_t) + \beta \int \nu(w', D', S_{t+1}^\gamma) dF(D', D) \right]
\]

subject to

- $c_t \geq 0, \psi_t(i) \in [0, B_t(i)], \ i = 1, 2, \ldots, m,$
- $c_t + a_t + \theta_t s_t(D) + \psi_t \cdot p_t(D) \leq w$
- $\theta_t(s_{t+1}(D') + (1 - q)D') + \psi_t \cdot (Ap_{t+1}(D') + b) = w'.$
The last equation

\[
\nu(w, D, \gamma \bar{I}) = \max \left[ u(c_n) + \phi(a_n - \gamma_n) + \beta \int \nu(w', D', \gamma \bar{I}) dF(D', D) \right]
\]

subject to

\[
\begin{align*}
    c_n &\geq 0, \psi_n(i) \in [0, B_n(i)], i = 1, 2, \ldots, m, \\
    c_n + a_n + \theta_n s_n(D) + \psi_n \cdot p_n(D) &\leq w \\
    \theta_n (s_n(D') + (1 - q)D') + \psi_n \cdot (Ap_n(D') + b) &= w'.
\end{align*}
\]
The uniqueness of the value function

The last equation determines $v(w, D, \gamma I)$ as a fixed point of a contraction. Then we can determine $v(w, D, S^t \gamma)$ consecutively using the first $n$ equations.
Assuming $u$ and $\phi$ are smooth and the maximum in dynamic programming are attained, we have $w \rightarrow v(w, D, \bar{\gamma})$ is differentiable and

$$v_w(w, D, \bar{\gamma}) = u'(c) = \phi'(a - \gamma_0).$$ (11)

Moreover $w \rightarrow v(w, D, \bar{\gamma})$ is an increasing concave function.
Definition of the equilibrium

Equilibrium

Let $\vec{\gamma}$ be a consumption commitment with $\gamma_t = 0$ for $t \geq n$. An equilibrium is a pair of stock and bond prices processes that $\vec{s}, \vec{p}$ and a continuous bounded optimal value function $V(\theta, \psi, D, \vec{\gamma}|\vec{s}, \vec{p})$ satisfying

(i) $S^t \vec{p} = S^{t'} \vec{p}, S^t \vec{s} = S^{t'} \vec{s}$, for $t, t' \geq n$,

(ii) dynamic programming system of equations, and

(iii) for each $D$ and $t = 0, 1, \ldots, n$, $V(1, B_t, D, S^t \vec{\gamma}|S^t \vec{s}, S^t \vec{p})$ is attained by $\theta' = 1, \psi' = B_t$, and

$$c + a = (1 - q)D + B_{t-1} \cdot (Ap_t(D) + b) - B_t \cdot p_t(D).$$
Using the properties of the optimal value function the optimality condition for the dynamic programming equations characterizes the equilibrium.
Equation systems characterizing the equilibrium

For $t = 0, 1, \ldots, n - 1,$

$$u'(c_t(D, p_t))(s_t(D), p_t(D))$$

$$= \beta \int u'(c_{t+1}(D', p_{t+1}))(s_{t+1}(D') + (1 - q)D', A p_{t+1}(D') + b) dF(D', D)$$

and

$$u'(c_n(D, p_n))(s_n(D), p_n(D))$$

$$= \beta \int u'(c_n(D', p_n))(s_n(D') + (1 - q)D', A p_n(D') + b) dF(D', D).$$
Existence and uniqueness

**Existence** is easier: impose condition so that the Schauder’s fixed point theorem applies to the equation system characterizing the equilibrium.

**Uniqueness** is harder: impose condition so that the Banach fixed point theorem applies to the equation system characterizing the equilibrium.

Modeling right those conditions are satisfied.
Computing the equilibrium

Banach’s iterative algorithm for contractions.
Parameters, utility and penalty functions

We use parameters that mimic the US bond markets: assuming 1 to 30 year bonds with heavy short term (1 year) liquidity. We use the following utility and penalty functions

\[ u(t) = \ln(t) \]

and

\[ \phi(t) = \frac{2}{k^2} t^2. \]
Without consumption commitment

Figure: Yield surface: no consumption commitment
Typical consumption commitments

Figure: Yield surface: consumption commitment at $t = 6$
Impact of one consumption commitment at 8: no addition supply of bonds

Figure: Deviation due to a single consumption commitment
Observations

- the yield of 8 year bond is much lower than other bonds due to the consumption commitment;
- the yields of the other bonds are also depressed,
- the yield reductions for bonds with maturity close to the 8 year bonds are reduced more;
- the yield of the bonds with very short maturity – 1,2,3 are heavily depressed – due to the formula
  \[ \frac{1}{i} \ln p_0(i) \]
  and
- the depression of the yield for bonds with maturity shorter than 8 are heavier than those with maturity longer than 8.
The role of liquidity

The effect we see in the previous picture is the combination of the heavy short term (1 year) liquidity and the liquidity reduction at the time of the commitment. Taking away these we have

Figure: Isolating the effect of consumption commitment
Conclusion

1. Consumption commitments at $t$ reduce the yields of the $t$ year bonds and those close to it.
2. The term structure is the result of multiple factors: consumption commitments, the status of the economy, and liquidity.
3. Higher comparative liquidity for $t$ year bonds causes a substitution effect of period $t$. Substitution effect we observe in real markets is due to the high liquidity of short term debt instruments.
4. Providing additional liquidity for a bond in unusually high demand by the government is an effective way of minimizing its impact.
Problems for further study

1. Nonsmooth utility and penalty functions.
2. Role of duality.
3. Relationship with other models.
4. Model real markets.