In Arrear Term Structure Products: No Arbitrage Pricing Bounds and The Convexity Adjustments

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Convexity adjustment (correction) occurs when the interest rate pays out at the wrong time and/or in the wrong currency.

In-arrear swaps and in-arrear caps and floors differ from vanilla products in the floating leg:

- In the vanilla products, the payoff at each time $t_i$ is based on the LIBOR rate observed at the date before, i.e. at $t_{i-1}$.

- In the in-arrear products, the payoff at each time $t_i$ is based on the LIBOR rate at $t_i$.

Compared to the vanilla products, there is mismatch in cash flow timing.
Valuation of in-arrear interest rate products

- Mostly when no further assumptions concerning the term structure of the interest rate are made, it is impossible to price these products.
- Even when a specific term structure model is assumed, closed-form solutions can only be achieved in very few cases (e.g. LIBOR market model).
- Approximation methods are frequently used:
  - The price of an in-arrear swap is usually approximated by the sum of a forward-starting vanilla swap and a convexity adjustment term.
  - Similarly, the price of an in-arrear cap/floor can be decomposed into a vanilla cap/floor plus a convexity adjustment.
Notations

Consider an in-arrear payer swap

Assume that the variable interest payments (floating leg) are received at a set of equidistant time points in

\[ T = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\} \]

with \( \alpha := t_{i+1} - t_i \).

The fixed leg pays fixed interest rate payments at a set of equidistant time points in

\[ \Theta = \{0 = \tau_0 < \tau_1 < \cdots \tau_n := T\} \]

Let \( \beta := \tau_{j+1} - \tau_j \). Assume that \( \beta \) is a multiple of \( \alpha \), i.e. \( \beta = m \cdot \alpha \), \( m \in \mathbb{N} \). It implies that \( \tau_n = t_{n \cdot m} = t_N = T \).
Payoff structure of an in-arrear payer swap

Payoff structure of an in-arrear payer swap with $m = 4$ and

$$r_L(t_i, t_i, \alpha) := r_L(t_i)$$
Accumulation of the variable interest rate from $t_i$ to $t_{i+1}$

$$\alpha \cdot r_L(t_i) \cdot V$$

Accumulation

$$\alpha \cdot r_L(t_i) \cdot V \cdot (1 + \alpha \cdot r_L(t_i))$$

$$= \alpha \cdot r_L(t_i) \cdot V + \alpha^2 \cdot r^2_L(t_i) \cdot V$$

variable interest payment of a vanilla swap

additional interest payments
The initial arbitrage-free value of the floating leg of the in-arrear swaps can be decomposed into two parts:

a) the initial arbitrage-free value of the floating leg of a forward-starting vanilla swap which starts to receive $\alpha \cdot V \cdot r_L(t_1)$ at time $t_2$ and ends with the last payment $\alpha \cdot V \cdot r_L(t_N)$ at time $t_{N+1} := t_N + \alpha$,

b) and the initial arbitrage-free value of a sequence of additional variable interest payments:

$$\alpha^2 \cdot r^2_L(t_i) \cdot V, \quad i = 2, \ldots, N + 1.$$
The payoff of the forward-starting vanilla swap can be duplicated by the following simple strategy:

- At time 0 buy \( V \) zero coupon bonds with maturity \( t_1 \).
- At time \( t_1 \), invest \( V \) as a rollover deposit with the time-varying LIBOR rate.
- At the end \( t_{i+1} \) of each period \([t_i, t_{i+1}]\), \( i = 1, \ldots, N \), keep the interest rate payment, and invest \( V \) further for one more period.
- Repeat this process until \( t_{N+1} := t_N + \alpha \). The terminal strategy value is \( V + \alpha \cdot V \cdot r_L(t_N) \).
- At time 0 sell \( V \) zero coupon bonds with maturity \( t_{N+1} \).

The initial arbitrage-free value of part a) of the floating leg corresponds to the value of the entire strategy, i.e.

\[
VB(t_0, t_1) - VB(t_0, t_{N+1})
\]
Impossible to determine the arbitrage-free values of this part without further assumptions about the term structure of the interest rate

One trick is to make the following approximation

\[ r_L(t_i, t_i, \alpha) := r_L(t_i) \approx r_L(t_0, t_i, \alpha) \]

The technique of replacing the random spot LIBOR rate by the forward LIBOR rate at time zero is the so-called convexity adjustments approach
Convexity adjustment for the initial arbitrage-free value

- The model-independent approximation for the initial arbitrage-free value of the in-arrear payer swap:

\[
payer\text{-}swap_{ar}[r_L, L, V, T, \Theta] \\
\approx V \cdot \left( B(t_0, t_1) - B(t_0, t_{N+1}) - \left( \sum_{j=1}^{n} \beta \cdot L \cdot B(\tau_0, \tau_j) \right) \right) \\
+ V \cdot \sum_{i=1}^{N} \left( \frac{B(t_0, t_i) - B(t_0, t_{i+1})}{B(t_0, t_{i+1})} \right)^2 \cdot \alpha r_L(t_0, t_i, \alpha)
\]
Assumption

Assumption:

Assume an arbitrage-free financial market with a set of equivalent martingale measures $\mathbb{P}$. It is furthermore assumed that there exists at least an equivalent martingale measure $P^* \in \mathbb{P}$ such that $P^*$ can be changed to a forward measure $Q^{\tau+\alpha}$ for each compounding period $\alpha$ and each time $\tau$ through the change-of-measure technique and under $Q^{\tau+\alpha}$ the forward LIBOR rate process $\{r_L(t, \tau, \alpha)\}_{t \leq \tau}$ is a martingale. It holds particularly

$$r_L(t_0, \tau, \alpha) = E_{Q^{\tau+\alpha}}[r_L(t, \tau, \alpha)].$$
Proposition

In any arbitrage-free interest rate model with the above assumption, the convexity adjustments of an in-arrear payer swap is a lower bound for the arbitrage-free price of an in-arrear payer swap:

\[
payer\text{-}swap_{ar}[r_L, L, V, T, \Theta] \geq A(L)
\]

where \( T = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\} \) with \( \alpha := t_{i+1} - t_i \) and \( \Theta = \{0 = \tau_0 < \tau_1 < \cdots \tau_n : T\} \) with \( \beta = \tau_{j+1} - \tau_j = m \cdot \alpha \). The pricing bound \( A(L) \) owns the form of

\[
A(L) = V \cdot B(t_0, t_1) - V \cdot B(t_0, t_{N+1}) + \sum_{i=1}^{N} V \cdot B(t_0, t_{i+1}) \left( \frac{B(t_0, t_i)}{B(t_0, t_{i+1})} - 1 \right)^2
\]

\[- \sum_{j=1}^{n} \beta \cdot L \cdot V \cdot B(t_0, t_{j \cdot m}).\]

In particular, \( L^* \) with \( A(L^*) = 0 \) is a lower bound for the swap yield of an in-arrear payer swap.
Proof Sketch

What needs to be approximated in the in-arrear swap is the sequence of additional variable interest rate payments $\alpha^2 \cdot r_L^2(t_j, t_j, \alpha)V, j = 1, \ldots, N$:

\[
EP^* \left[ e^{-\int_0^{t_{j+1}} r_u du} \alpha^2 r_L^2(t_j, t_j, \alpha)V \bigg| \mathcal{F}_t \right] \\
= B(t_0, t_{j+1})\alpha^2 V \mathbb{E}_{Q_{t_{j+1}}} \left[ r_L^2(t_j, t_j, \alpha) \big| \mathcal{F}_t \right] \\
= B(t_0, t_{j+1})\alpha^2 V \left[ (\mathbb{E}_{Q_{t_{j+1}}} [r_L(t_j, t_j, \alpha) \big| \mathcal{F}_t])^2 + \text{Var}_{Q_{t_{j+1}}} [r_L(t_j, t_j, \alpha) \big| \mathcal{F}_t] \right] \\
\geq B(t_0, t_{j+1})\alpha^2 V \left( \mathbb{E}_{Q_{t_{j+1}}} [r_L(t_j, t_j, \alpha) \big| \mathcal{F}_t] \right)^2 \\
= B(t_0, t_{j+1})\alpha^2 V r_L^2(t_0, t_j, \alpha) \\
= B(t_0, t_{j+1})V \left( \frac{B(t_0, t_j)}{B(t_0, t_{j+1})} - 1 \right)^2.
\]
Normal term structure of the interest rate, particularly that the log-rate of return of the zero coupon bonds are linearly increasing in time to maturity:

\[ y(t_0, t_i) = 2.5\% + 0.2\% \cdot (t_i - t_0). \]

Consequently, zero coupon bonds have the value of

\[ B(t_0, t_i) = \exp\{-y(t_0, t_i)(t_i - t_0)\}. \]

Other parameters are chosen as follows:

\[ \alpha = \frac{1}{4}, \beta = \frac{1}{2}, L = 3.5\%, V = 1 \]

We adopt the following function (c.f. Brigo and Mercurio (2001))

\[ \gamma(t, t_i) = [g + a(t_i - t)] \exp\{-b(t_i - t)\} + c \]

with \( a = 0.19085664; b = 0.97462314; c = 0.08089168; g = 0.01344948 \)
## Numerical results

<table>
<thead>
<tr>
<th>Maturity ( T ) in years</th>
<th>In-Arrear Payer Swap</th>
<th>LIBOR model</th>
<th>Convexity correction</th>
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<td>imp. ( L^* )</td>
<td>Arb-free price</td>
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Arbitrage-free prices for (in-arrear) payer swaps and implied swap yield based on exact pricing formulae and the corresponding convexity adjustments (pricing bounds).
As the reference payment time, we fix again an equidistant sequence of time points \( T = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\} \) with \( \alpha := t_{i+1} - t_i \), and \( L \) is now the cap/floor rate.

In a vanilla cap, the contract holder receives \( \alpha \cdot V \cdot [r_L(t_i) - L]^+ \) at \( t_{i+1}, \ i = 1, \cdots, N \). Hereby we have used \( [x]^+ := \max\{x, 0\} \).

The contract holder of an in-arrear cap receives

\[
\alpha \cdot V \cdot [r_L(t_i) - L]^+
\]

already at time \( t_i, \ i = 1, \cdots N \).
Payoff structure of an in-arrear cap

\[\begin{align*}
\text{At } t_0 &: \alpha \cdot V \cdot [r_L(t_1) - L]^+ \\
\text{At } t_1 &: \alpha \cdot V \cdot [r_L(t_2) - L]^+ \\
\text{At } t_2 &: \alpha \cdot V \cdot [r_L(t_3) - L]^+ \\
\text{At } t_3 &: \alpha \cdot V \cdot [r_L(t_4) - L]^+ \\
\text{At } t_4 &: \alpha \cdot V \cdot [r_L(t_{N-1}) - L]^+ \\
\text{At } t_{N-1} &: \alpha \cdot V \cdot [r_L(t_N) - L]^+
\end{align*}\]
Accumulation of the payment from $t_i$ to $t_{i+1}$

$\alpha \cdot V \cdot [r_L(t_i) - L]^+$

$\frac{(1 + \alpha \cdot r_L(t_i)) \cdot \alpha \cdot V \cdot [r_L(t_i) - L]^+}{(1 + \alpha \cdot r_L(t_i))}$
Approximation

- Apparently, without further assumptions about the term structure of the interest rate, it is impossible to calculate the initial arbitrage-free value of the in-arrear cap.

- Use the initial forward LIBOR rate $r_L(t_0, t_i, \alpha)$ to approximate $r_L(t_i)$:

\[
(1 + \alpha \cdot r_L(t_i)) \approx (1 + \alpha \cdot r_L(t_0, t_i, \alpha)) = \frac{B(t_0, t_i)}{B(t_0, t_{i+1})}.
\]

- **Approximation** for the initial market value of the in-arrear cap:

\[
Cap_{ar}[r_L, L, V, \overline{T}, t_0] \approx \sum_{i=1}^{N} \frac{B(t_0, t_i)}{B(t_0, t_{i+1})} \cdot Caplet(r_L, L, V, t_0, t_{i+1})
\]

where $Caplet(r_L, L, V, t_0, t_{i+1})$ gives the initial arbitrage-free value of $\alpha \cdot V \cdot [r_L(t_i) - L]^+$ which becomes due at time $t_{i+1}$. 
Proposition

In an arbitrage-free interest rate model with the above mentioned assumption, there exists a model-independent lower bound for the arbitrage-free price of an in-arrear cap, i.e.

$$
Cap_{ar}[r_L, L, V, T, t_0] \geq \sum_{i=1}^{N} \frac{B(t_0, t_i)}{B(t_0, t_{i+1})} \cdot \text{Caplet}(r_L, L, V, t_0, t_{i+1})
$$

where $T = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\}$ with $\alpha := t_{i+1} - t_i$ and Caplet is defined by

$$
\text{Caplet}(r_L, L, V, t_0, t_{i+1}) := B(t_0, t_{i+1}) E_{Q^{t_{i+1}}}[^{\alpha} \cdot V \cdot [r_L(t_i) - L]^+] | F_{t_0}].
$$
Proof sketch

\[ Cap_{ar}[r_L, L, V, T, t_0] \]

\[ = \sum_{i=1}^{N} E_{P^*} \left[ e^{-\int_{0}^{t_{i+1}} r_u du} (1 + \alpha \cdot r_L(t_i)) \cdot \alpha \cdot V \cdot [r_L(t_i) - L]^+ \big| \mathcal{F}_{t_0} \right] \]

\[ = \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}} \left[ (1 + \alpha \cdot r_L(t_i)) \cdot \alpha \cdot V \cdot [r_L(t_i) - L]^+ \big| \mathcal{F}_{t_0} \right] \]

\[ = \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}} \left[ \alpha \cdot V \cdot [r_L(t_i) - L]^+ \big| \mathcal{F}_{t_0} \right] \]

\[ + \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}} \left[ \alpha^2 \cdot V \cdot r_L(t_i) \cdot [r_L(t_i) - L]^+ \big| \mathcal{F}_{t_0} \right] \]

\[ = \sum_{i=1}^{N} \text{Caplet}(r_L, L, V, t_0, t_{i+1}) + \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}} \left[ \alpha^2 \cdot V \cdot r_L(t_i) \cdot [r_L(t_i) - L]^+ \big| \mathcal{F}_{t_0} \right] \]
...Proof sketch

The second term on the right-hand side can be rewritten as

\[
\sum_{i=1}^{N} B(t_0, t_{i+1}) E_{Q^{t_{i+1}}} \left[ \alpha^2 \cdot V \cdot r_L(t_i) \cdot \left[ r_L(t_i) - L \right]^+ \bigg| \mathcal{F}_{t_0} \right]
\]

\[
= \alpha^2 \cdot V \cdot \sum_{i=1}^{N} B(t_0, t_{i+1}) E_{Q^{t_{i+1}}} \left[ \left[ r_L(t_i) - L \right]^+ \bigg| \mathcal{F}_{t_0} \right] \cdot E_{Q^{t_{i+1}}} \left[ r_L(t_i) \big| \mathcal{F}_{t_0} \right]
\]

\[
+ \alpha^2 \cdot V \cdot \sum_{i=1}^{N} B(t_0, t_{i+1}) \text{Cov}_{Q^{t_{i+1}}} \left[ r_L(t_i), \left[ r_L(t_i) - L \right]^+ \bigg| \mathcal{F}_{t_0} \right]
\]

\[
\geq \alpha^2 \cdot V \cdot \sum_{i=1}^{N} B(t_0, t_{i+1}) E_{Q^{t_{i+1}}} \left[ \left[ r_L(t_i) - L \right]^+ \bigg| \mathcal{F}_{t_0} \right] \cdot E_{Q^{t_{i+1}}} \left[ r_L(t_i) \big| \mathcal{F}_{t_0} \right]
\]

\[
= \sum_{i=1}^{N} \alpha \cdot r_L(t_0, t_i, \alpha) \cdot B(t_0, t_{i+1}) \cdot \alpha \cdot V E_{Q^{t_{i+1}}} \left[ \left[ r_L(t_i) - L \right]^+ \bigg| \mathcal{F}_{t_0} \right]
\]
### Numerical results

<table>
<thead>
<tr>
<th>Maturity $T$ in years</th>
<th>Cap contract</th>
<th>Floor contract</th>
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Arbitrage-free prices for (in-arrear) caps and floors based on exact pricing formulae and the corresponding convexity adjustments (pricing bounds).
Comparison of the convexity adjustment and the exact value

|Exact value - Convexity adjustment|/Exact value in percentage with parameters: $\alpha = \frac{1}{4}$, $T = 5$.}
In the present paper, we provide a strong theoretical argument to support the convexity adjustments approach, a rule of thumb used by practitioners to value in-arrear products.

Our results exclusively depend on the no-arbitrage condition which ensures the existence of certain forward risk adjustment measures. They are model-independent and in effect give pricing bounds for in-arrear term structure products.