Discrete-Time Minimum-Variance Hedging of European Contingent Claims

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Derivative: A financial asset which derives its present value from the uncertain future value of an underlying risky asset

- Pricing problem: To determine a fair present value of the derivative
- Hedging problem: To determine a trading strategy to minimize the seller’s risk

Black and Scholes (1973), Merton (1973): If the underlying asset price follows a geometric Brownian motion (GBM), then

- There exists a self-financed hedging portfolio that replicates the derivative
- The initial investment on the portfolio gives the unique fair price of the derivative

**Catch:** The replicating strategy requires trading in continuous time
In practice, observations and trades possible only at discrete times
Transaction costs make frequent trading potentially expensive
Exact replication and complete elimination of risk not possible with discrete-time trading
Look for strategies to minimize various measures of risk
Apply strategy to a particular market model such as GBM
  - Angelini and Herzel (2009): Minimum-variance hedging for a European call option
**Our Objective:** To extend to general European-type derivatives including path dependent options
Discrete-Time Hedging

\[ T_0 = 0 \quad T_1 \quad T_{n-1} \quad T_n = T \]

Discrete-time hedging

Discounted cash flows

Premium

\[ \Delta_1 \bar{S}_0 \quad \Delta_1 \bar{S}_1 \]

\[ \Delta_2 \bar{S}_1 \]

Payoff \( \bar{V} \)

\[ \Delta_{n-1} \bar{S}_{n-1} \quad \Delta_n \bar{S}_n \]

\[ \Delta_n \bar{S}_{n-1} \]
The Minimum-Variance Hedging Problem

- **Problem:** Determine the positions $\Delta_1, \ldots, \Delta_n$ in terms of available information such that the risk-neutral variance of the final money position is minimized.

- **Assumption:** All relevant random variables are square integrable.

- **Solution:** (Föllmer and Schweizer (1989), Schäl (1994))

\[
\Delta^*_k = \frac{\text{covar}(S_k, V_k|\mathcal{F}_{k-1})}{\text{var}(S_k|\mathcal{F}_{k-1})}
\]

- $\mathcal{F}_k = \sigma$-algebra generated by underlying asset prices observed up to $T_k$.
- $V_k$ = arbitrage-free price of ECC at $T_k$.

A model for the underlying asset price process is required for computing the strategy.
Minimum-Variance Hedging for GBM

\[ dS_t = rS_t dt + \sigma S_t dW_t \]
\[ S_t = S_0 \exp \left[ (r - \frac{1}{2} \sigma^2) t + \sigma W_t \right] \]

- Need to compute \( \mathbb{E}(S_k V_k | F_{k-1}) \)
- In case of simple claims
  - \( V_k \) is an explicit function \( v(T_k, S_k) \) of \( S_k \),
  - Given \( S_{k-1}, S_k \) is log normal

\[
\mathbb{E}(S_k V_k | F_{k-1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(T_k, S_{k-1} e^x) S_{k-1} e^x e^{-\frac{(x-\mu)^2}{2\Lambda^2}} \, dx
\]

- For a general path-dependent claim
  - \( V_k \) may depend on other path-dependent variables
  - The joint distributions of those variables with \( S_k \) will be needed
Preliminaries on the Wiener Space

- **Wiener space** $\Omega$ = linear vector space of continuous functions on $[0, T]$ zero at $t = 0$, equipped with topology of uniform convergence
- **Wiener measure** $P$, the “distribution” of the Wiener process
- Coordinate process $W(t, \omega) = \omega(t)$ is a Wiener process
- **Cameron-Martin Theorem**
  - Suppose $h \in \Omega$ is absolutely continuous and $\dot{h}$ is square integrable
  - Define $\tau^h : \Omega \to \Omega$ by $\tau^h(\omega) = \omega + h$
  - Let $P^h = \text{push-forward of } P \text{ by } \tau^h$, that is, $P^h(A) = P(\tau^{-h}(A))$
  - Then $\frac{dP^h}{dP} = \exp \left[ \int_0^T \dot{h}(s) dW_s - \frac{1}{2} \int_0^T \dot{h}^2(s) ds \right]$
  - $\mathbb{E} \left( X \frac{dP^h}{dP} \middle| \mathcal{F}_t \right) = \mathbb{E}^{P^h}(X|\mathcal{F}_t) \mathbb{E} \left( \frac{dP^h}{dP} \middle| \mathcal{F}_t \right)$
  - If $\dot{h} \equiv 0$ on $[t, T]$, then $\mathbb{E}^{P^h}(X|\mathcal{F}_t) = \mathbb{E}(X|\mathcal{F}_t)$
  - If $h \equiv 0$ on $[0, t]$, then $\mathbb{E}^{P^h}(X|\mathcal{F}_t) = \mathbb{E}(X \circ \tau^h|\mathcal{F}_t)$
Minimum-Variance Hedging for GBM

To find $\mathbb{E}(V_k S_k | \mathcal{F}_{k-1})$

$$S_k = S_{k-1} \exp \left[ (r - \frac{1}{2} \sigma^2) (T_k - T_{k-1}) + \sigma (W_k - W_{k-1}) \right]$$

$$= S_{k-1} e^{r \delta_k} \exp \left[ \int_{0}^{T} \dot{h}_k(s) dW_s - \frac{1}{2} \int_{0}^{T} \dot{h}_k^2(s) ds \right] = S_{k-1} e^{r \delta_k} \frac{dP^{h_k}}{dP}$$

$$h_k(t) \overset{\Delta}{=} \int_{0}^{t} \sigma \chi_{[T_{k-1}, T_k]}(s) ds, \quad \delta_k \overset{\Delta}{=} T_k - T_{k-1}$$

$$\mathbb{E}(V_k S_k | \mathcal{F}_{k-1}) = e^{r \delta_k} e^{-r (T - T_k)} S_{k-1} \mathbb{E}(V \circ \tau^{h_k} | \mathcal{F}_{k-1})$$

$$\Delta^*_k = \frac{e^{-r (T - T_{k-1})}}{(e^{\sigma^2 \delta_k} - 1) S_{k-1}} \mathbb{E}(V \circ \tau^{h_k} - V | \mathcal{F}_{k-1})$$
Some Observations

- Numerator (denominator) = conditional expectation of change in payoff (asset price) when Wiener paths are shifted by the function $h_k$
- Extends to time-varying (but deterministic) volatility
- If $V$ is a functional of the stock path, then $V \circ \tau^{h_k}$ is the same functional of the modified asset-price path $\hat{S}_t \triangleq e^{\sigma h_k(t)} S_t$ satisfying
  \[ d\hat{S}_t = (r + \sigma \dot{h}_k(t))\hat{S}_t dt + \sigma \hat{S}_t dW_t \]
- Modified asset-price path follows a GBM with time-varying (but deterministic) interest rate
- A closed-form pricing solution for the modified GBM will yield a closed-form solution for the minimum-variance hedging strategy
Example: Simple Claims

- Payoff determined solely by underlying asset price at maturity
- If \( V = \phi(S_T) \), then \( V \circ \tau^h_k = \phi(e^{\sigma^2 \delta_k} S_T) \)
- ECC price at time \( t \) determined solely by \( S_t \)
- If \( v(t, S_t) = e^{-r(T-t)} \mathbb{E}(V | \mathcal{F}_t) \), then \( e^{-r(T-T_{k-1})} \mathbb{E}(V \circ \tau^h_k | \mathcal{F}_{k-1}) = v(t, e^{\sigma^2 \delta_k} S_{k-1}) \)

\[
\Delta^*_k = \frac{v(T_{k-1}, e^{\sigma^2 \delta_k} S_{k-1}) - v(T_{k-1}, S_{k-1})}{(e^{\sigma^2 \delta_k} - 1) S_{k-1}}
\]
Some More Observations

- If the price has a closed-form solution then the minimum variance hedging strategy is easily computable.
- In contrast, the delta-neutral needs differentiability of the pricing function.
- Monte-Carlo can be used in general.
- Expand $\Delta_k^*$ using Taylor series in $\delta_k$

$$
\Delta_k^* = \frac{\partial v}{\partial x}(T_{k-1}, S_{k-1}) + \frac{\sigma^2 \delta_k}{2} S_{k-1} \frac{\partial^2 v}{\partial x^2}(T_{k-1}, S_{k-1}) + o(\delta_k^2)
$$

- Variance approaches zero as hedging frequency increases, perfect replication achieved in the limit.
Example: Asian-type Claims

- Payoff determined by underlying asset price and its continuously sampled geometric average at maturity

\[ V = \phi(S_T, G_T), \quad G_t \triangleq \exp \left[ \frac{1}{T} \int_0^t \log S_u \, du \right] S_t^{(T-t)/T} \]

\[ V \circ \tau^{h_k} = \phi(e^{\sigma^2 \delta_k} S_T, e^{\sigma^2 \eta_k} G_T), \quad \eta_k \triangleq \frac{\delta_k}{T} \left[ T - \frac{1}{2}(T_k + T_{k-1}) \right] \]

- ECC price at \( t \) determined by \( S_t \) and \( G_t \)

- If \( v(t, S_t, G_t) = e^{-r(T-t)} \mathbb{E}(V|\mathcal{F}_t) \), then

\[ e^{-r(T-T_{k-1})} \mathbb{E}(V \circ \tau^{h_k} | \mathcal{F}_{k-1}) = v(t, e^{\sigma^2 \delta_k} S_{k-1}, e^{\sigma^2 \eta_k} G_{k-1}) \]

\[ \Delta^*_k = \frac{v(T_{k-1}, e^{\sigma^2 \delta_k} S_{k-1}, e^{\sigma^2 \eta_k} G_{k-1}) - v(T_{k-1}, S_{k-1}, G_{k-1})}{(e^{\sigma^2 \delta_k} - 1) S_{k-1}} \]
Summary and Ongoing Work

Minimum-variance hedging strategy for a path-dependent ECC
- Involves pricing the ECC when Wiener paths are shifted
- Easy to implement numerically
- Can be expressed in terms of pricing functions
- Closed-form expressions possible for loglinear claims
  - Simple, Asian-Geometric, Cliquet, Multi-Look Options

Possible extensions
- Multi-asset options such as exchange and basket options
- Options involving random exercise or knock out
- Stochastic volatility models
- Hedging using a portfolio of derivative assets