Perpetual Cancellable Call Option

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Outline

Game Options
   Complete Market Valuation
   Optimal Policies

Perpetual Cancellable Call Option
   Previous results: Perpetual Cancellable Put Option
   Valuation
   Conclusions
The current financial crisis has highlighted the importance of adequately hedging risk and limiting downside losses.

Hedging:
- Modeling fluctuations in value under changes in market factors ($X$, $\sigma$, etc) and constructing offsetting positions in tradable assets.

Another way is to build extra features, such as cancellation, into the derivative specifications.
Consider an American-style derivative with a cancellation feature given to the writer of the contract.

At any point during the life of the contract, the writer can force the holder to take the current payoff plus a small additional amount as compensation for terminating the contract.
Game Option

- Consider an American-style derivative with a cancellation feature given to the writer of the contract.

- At any point during the life of the contract, the writer can force the holder to take the current payoff plus a small additional amount as compensation for terminating the contract.

- We refer to this as a Game option.
Game Option

- Contract: Seller $A$  Buyer $B$

- $B$ can *exercise* at any time $t$.
  \[ A \xrightarrow{Y_t} B \]

- $A$ can *cancel* at any time $t$.
  \[ A \xrightarrow{Y_t+\delta_t} B \]

- Two optimal stopping problems: Optimal Exercise Time and Optimal Cancellation Time.

- What is the fair price $V$ that $B$ should pay to $A$ for the contract? What are the optimal exercise and cancellation times for this game option?
In this setting, valuation corresponds to solving a zero-sum optimal stopping game between two players.

For cancellation policy $\tau$ and exercise policy $\sigma$ the payoff of the claim is

$$R(\sigma, \tau) := (Y_{\tau} + \delta_{\tau})1_{\{\tau < \sigma\}} + Y_\sigma 1_{\{\sigma \leq \tau\}}$$
Valuation: Complete Markets

- In this setting, valuation corresponds to solving a zero-sum optimal stopping game between two players.

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$$R(\sigma, \tau) := (Y_\tau + \delta_\tau)1_{\{\tau < \sigma\}} + Y_\sigma 1_{\{\sigma \leq \tau\}}$$

- Kifer (2000) shows the fair price is

$$V_t = \inf_{\tau \in S_t, T} \sup_{\sigma \in S_t, T} \mathbb{E}[e^{-r(\sigma \land \tau - t)} R(\sigma, \tau) | F_t]$$

$$= \sup_{\sigma \in S_t, T} \inf_{\tau \in S_t, T} \mathbb{E}[e^{-r(\sigma \land \tau - t)} R(\sigma, \tau) | F_t]$$

- Basic Price Bound: $Y_t \leq V_t \leq Y_t + \delta_t$. 

Optimal Policies

- What are the ‘optimal’ $\sigma$, $\tau$ stopping times?

i.e. What policies achieve the infimum and supremum?

$$\sigma_t^* := \inf \{s \geq t : V_s = Y_s\}$$
$$\tau_t^* := \inf \{s \geq t : V_s = Y_s + \delta_s\}$$

- These stopping times are also ‘optimal’ exercise dates.
  - The holder waits until the value drops to the exercise value.
  - The writer waits until the value reaches the cancellation value.
Perpetual Cancellable Call Option

- Let the risky asset $X$ satisfy the following risk-neutralized evolution

$$dX_t = (r - d)X_t \, dt + \sigma X_t \, dW_t$$

- Suppose $T = \infty$ and consider the following:

$$Y_t = (X_t - K)^+; \quad \delta_t = \delta > 0$$

- We call this a *Perpetual Cancellable Call Option* or simply a *$\delta$-penalty call option*. 
Valuation of $\delta$-penalty Put Option

Completed by Kyprianou (2004):

- Value function identified explicitly.
- Optimal Stopping times (for $\delta$ small):

$$\sigma^* := \inf \{ t \geq 0 : X_t = k^* \}$$
$$\tau^* := \inf \{ t \geq 0 : X_t = K \}$$

Does the valuation of a Call Option with dividend $d > 0$ follow symmetrically to this result?
Optimal Policies for Perpetual American Call?

**Figure:** Possible Exercise and Cancellation Barriers
Valuation: \( r \leq d \)

**Conjecture:** Value function satisfies for \( x \in (0, K) \),

\[
\mathcal{L} V - r V = 0
\]

\[
V(K) = \delta, \quad \lim_{x \downarrow 0} V(x) = 0
\]

and for \( x \in (K, k^*) \),

\[
\mathcal{L} V - r V = 0
\]

\[
V(K) = \delta, \quad V(k^*) = (k^* - K)^+, \quad V_x(k^*) = 1,
\]

where

\[
\mathcal{L} := (r - d)x \frac{d}{dx} + \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2}
\]
Valuation: \( r \leq d \)

**Proof:** Let \( v(x) \) be the proposed value function.

\[
v(x) \leq \inf_{\tau \in S_{0,\infty}} \mathbb{E}_x \left[ e^{-r(\tau \wedge \sigma_k^*)} v(X_{\tau \wedge \sigma_k^*}) \right]
\]

\[
\leq \inf_{\tau \in S_{0,\infty}} \mathbb{E}_x \left[ e^{-r(\tau \wedge \sigma_k^*)} (X_{\sigma_k^*} - K)^+ 1\{\sigma_k^* \leq \tau\} \right] 
+ ((X_\tau - K)^+ + \delta) 1\{\tau < \sigma_k^*\})]
\]

\[
\leq \sup_{\sigma \in S_{0,\infty}} \inf_{\tau \in S_{0,\infty}} \mathbb{E}_x \left[ e^{-r(\tau \wedge \sigma)} (X_{\sigma} - K)^+ 1\{\sigma \leq \tau\} \right] 
+ ((X_\tau - K)^+ + \delta) 1\{\tau < \sigma\})]
\]

\[
\leq \sup_{\sigma \in S_{0,\infty}} \mathbb{E}_x \left[ e^{-r(\tau_k \wedge \sigma)} (X_{\sigma} - K)^+ 1\{\sigma \leq \tau_k\} \right] 
+ ((X_{\tau_k} - K)^+ + \delta) 1\{\tau_k < \sigma\})]
\]

\[
\leq \sup_{\sigma \in S_{0,\infty}} \mathbb{E}_x \left[ e^{-r(\tau_k \wedge \sigma)} v(X_{\tau_k \wedge \sigma}) \right] 
\leq v(x)
\]
Value function: $r \leq d$

Conclusion:

$$V(x) = \begin{cases} 
  x - K & \text{if } x \in [k^*, \infty) \\
  g(x) & \text{if } x \in (K, k^*) \\
  \delta \left( \frac{x}{K} \right)^{\frac{\lambda}{\sigma} - \kappa} & \text{if } x \in (0, K) 
\end{cases}$$

$$g(x) := (k^* - K) \left( \frac{k^*}{x} \right)^\kappa \frac{(K/x) - \frac{\lambda}{\sigma} - (K/x)^{\frac{\lambda}{\sigma}}}{(k^*/K)^{\frac{\lambda}{\sigma}} - (k^*/K)^{-\frac{\lambda}{\sigma}}} + \delta \left( \frac{K}{x} \right)^\kappa \frac{(k^*/x)^{\frac{\lambda}{\sigma} - (k^*/x)^{-\frac{\lambda}{\sigma}}}}{(k^*/K)^{\frac{\lambda}{\sigma}} - (k^*/K)^{-\frac{\lambda}{\sigma}}}$$
Value function $r \leq d$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{convex_value_function.png}
\caption{Convex value function.}
\end{figure}
Valuation: $r > d$

Conjecture: Why not same as before?
Valuation: $r > d$

**Conjecture:** Why not same as before?
- $v(x)$ violates basic inequality
  
  $$ (x - K)^+ \leq v(x) \leq (x - K)^+ + \delta $$

**Figure:** $v(x)$ (dark blue) violates upper bound.
Valuation: $r > d$

**New Conjecture:** Value function satisfies

for $x \in (0, K)$,

$$\mathcal{L} V - rV = 0$$

$$V(K) = \delta, \quad \lim_{x \downarrow 0} V(x) = 0$$

and for $x \in (h^*, k^*)$,

$$\mathcal{L} V - rV = 0,$$

$$V(h^*) = (h^* - K)^+ + \delta, \quad V_x(h^*) = 1,$$

$$V(k^*) = (k^* - K)^+, \quad V_x(k^*) = 1$$
Valuation: $r > d$

Proof:

$v(x) \geq \sup_{\sigma \in {S_0, \infty}} \mathbb{E}_x [e^{-r(\sigma \wedge \tau_{[K, h^*]})} \{((X_{\tau_{[K, h^*]}^\tau} - K)^{+} + \delta)1\{\tau_{[K, h^*]} < \tau\}] + (X_\sigma - K)^{+}1\{\sigma \leq \tau_{[K, h^*]}\}]$

$\geq \inf_{\tau \in {S_0, \infty}} \sup_{\sigma \in {S_0, \infty}} \mathbb{E}_x [e^{-r(\sigma \wedge \tau)} \{((X_{\tau} - K)^{+} + \delta)1\{\tau < \sigma\}] + (X_\sigma - K)^{+}1\{\sigma \leq \tau\}]$

$\geq \sup_{\sigma \in {S_0, \infty}} \inf_{\tau \in {S_0, \infty}} \mathbb{E}_x [e^{-r(\sigma \wedge \tau)} \{((X_{\tau} - K)^{+} + \delta)1\{\tau < \sigma\}] + (X_\sigma - K)^{+}1\{\sigma \leq \tau\}]$

$\geq v(x)$

where $\tau_{[K,k^*]} := \inf\{ t \geq 0 : K \leq X_t \leq h^*\}$. 
Value function: $r > d$

Conclusion:

$$V(x) = \begin{cases} 
  x - K & \text{if } x \in [k^*, \infty) \\
  (k^* - K)^+ & \mathbb{E}_x[e^{-r\sigma_{k^*}}1\{\sigma_{k^*} \leq \tau_{[K, h^*]}\}] \\
  + ((h^* - K)^+ + \delta) & \mathbb{E}_x[e^{-r\tau_{[K, h^*]}1\{\tau_{[K, h^*]} < \sigma_{k^*}\}}] \\
  (x - K) + \delta & \text{if } x \in (h^*, k^*) \\
  \delta & \mathbb{E}_x[e^{-r\tau_{[K, h^*]}}] \\
  \delta & \mathbb{E}_x[e^{-r\tau_{[K, h^*]}}] \\
\end{cases}$$

where

$$\tau_{[K, k^*]} := \inf\{t \geq 0 : K \leq X_t \leq h^*\}$$

$$\sigma_{k^*} := \inf\{t \geq 0 : X_t \geq k^*\}$$
Value function $r > d$

**Figure:** Non-convex value function.
Some Implications

- Game Options with convex underlying payoffs are not necessarily convex.
- Subsequently, game option prices are not always increasing in the volatility parameter $\sigma$.

i.e., Vega can be negative,

$$\frac{\partial V(x)}{\partial \sigma} < 0, \text{ for some } x \text{ values.}$$
Thank You!

Thank you very much for your attention!
Some References


