Three make a dynamic smile
unspanned skewness and interacting volatility components in option valuation

Peter H. Gruber¹, Claudio Tebaldi² and Fabio Trojani¹

June 25, 2010
6th World Congress of the Bachelier Finance society, Toronto, Canada

¹ Università della Svizzera Italiana, Lugano, ² Università L. Bocconi, Milano
Introduction
Three questions

- How many sources of dynamic risk can we identify in index options?  
  *What is the empirical evidence?*

- How can we conveniently model the multiple risk sources in an affine framework?  
  *And thus account for the empirical evidence?*

- How can we improve on existing benchmark models?  
  *And put the model to an empirical test?*
DPS-type affine models

Duffie, Pan, Singleton (DPS, 2000): Transform analysis and asset pricing for affine jump-diffusions


Affine models with matrix jump diffusions

Leippold, Trojani (wp, 2008): Asset pricing with Matrix Jump Diffusions


Gourieroux, Sufana (wp 2004): Derivative Pricing with Multivariate Stochastic Volatility: Application to Credit Risk

da Fonseca, Grasselli, Tebaldi (2008): A Multifactor Volatility Heston Model

Alternative (affine) multifactor models

Muhle-Karb, Pfaffel, Stelzer (wp, 2010): Option pricing in multivariate stochastic volatility models of OU type

Empirical evidence
### Data: Call Options on the SP500 index

<table>
<thead>
<tr>
<th>Sample</th>
<th>calls only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time frame</td>
<td>1996-Sept/2008</td>
</tr>
<tr>
<td>Sampling interval</td>
<td>daily</td>
</tr>
<tr>
<td>Trading days</td>
<td>3205</td>
</tr>
<tr>
<td>Total number of observations</td>
<td>546'971</td>
</tr>
<tr>
<td>Average time to maturity</td>
<td>145 days [10d ~ 1yr]</td>
</tr>
<tr>
<td>Average moneyness ($S/K$)</td>
<td>1.05</td>
</tr>
<tr>
<td>Data processing</td>
<td>Bakshi(1997), no cuts</td>
</tr>
</tbody>
</table>
Data: Call Options on the SP500 index

Sample calls only

<table>
<thead>
<tr>
<th>Time frame</th>
<th>1996-Sept/2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling interval</td>
<td>daily</td>
</tr>
<tr>
<td>Trading days</td>
<td>3205</td>
</tr>
<tr>
<td>Total number of observations</td>
<td>546'971</td>
</tr>
<tr>
<td>Average time to maturity</td>
<td>145 days [10d ∼ 1yr]</td>
</tr>
<tr>
<td>Average moneyness ((S/K))</td>
<td>1.05</td>
</tr>
<tr>
<td>Data processing</td>
<td>Bakshi(1997), no cuts</td>
</tr>
</tbody>
</table>

Analytical framework: economically significant factors

- level \(V_t\): \(IV(\text{ATM}, \tau = 30d)\)
- skew \(S_t\): \([IV(\Delta = 0.4) - IV(\Delta = 0.6)] \cdot \frac{1}{(0.4-0.6)}\)
- term struct. \(M_t\): \([IV(\tau = 90d) - IV(\tau = 30d)] \cdot \frac{360}{(90-30)}\)
Skewness and term structure unconditionally highly correlated to level $\rightarrow$ level masks more nuanced effects.
Considerable variation in skewness and term structure that is not spanned by the volatility level.
Standard two-factor affine models cannot capture both unspanned skewness and term structure components.
Empirical evidence – factors

Principal component analysis (% of variance explained)

<table>
<thead>
<tr>
<th></th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional</td>
<td>96.8</td>
<td>1.9</td>
<td>0.9</td>
<td>0.1</td>
</tr>
</tbody>
</table>

$T = 3206$
Empirical evidence – factors

Principal component analysis (% of variance explained)

<table>
<thead>
<tr>
<th>$l$</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional</td>
<td>2</td>
<td>96.8</td>
<td>1.9</td>
<td>0.9</td>
<td>0.1</td>
</tr>
<tr>
<td>$0.08 &lt; V_t \leq 0.13$</td>
<td>3</td>
<td>84.5</td>
<td>6.9</td>
<td>5.6</td>
<td>0.8</td>
</tr>
<tr>
<td>$0.13 &lt; V_t \leq 0.17$</td>
<td>3</td>
<td>84.8</td>
<td>7.1</td>
<td>5.9</td>
<td>0.7</td>
</tr>
<tr>
<td>$0.17 &lt; V_t \leq 0.2$</td>
<td>3</td>
<td>75.4</td>
<td>12.3</td>
<td>8.4</td>
<td>1.3</td>
</tr>
<tr>
<td>$0.20 &lt; V_t \leq 0.23$</td>
<td>3</td>
<td>74.7</td>
<td>12.0</td>
<td>8.6</td>
<td>1.77</td>
</tr>
<tr>
<td>$0.23 &lt; V_t \leq 0.54$</td>
<td>3</td>
<td>87.2</td>
<td>8.7</td>
<td>2.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

$l =$ significant components according to mean eigenvalue criterion. ($N = 56$, threshold=$\frac{1}{56} = 1.79\%$)
Model
How to add a third factor

Bates-like independent factors – $SV(J)_{3,0}$

\[
\frac{dS_t}{S_t} = (r - q - \lambda_t \bar{k})dt + \sqrt{v_{1t}}dz_{1t} + \sqrt{v_{2t}}dz_{2t} + \sqrt{v_{3t}}dz_{3t} + k dN_t
\]

(1)

\[
dv_{it} = (\alpha_i - \beta_i v_{it}) dt + \sigma_i \sqrt{v_{it}} dw_{it} \quad i = 1, 2, 3
\]

(2)

Affine Matrix Jump Diffusion – $SV(J)_{3,1}$

\[
\frac{dS_t}{S_t} = (r - q - \lambda_t \bar{k})dt + \text{tr}(\sqrt{X_t} dz_t) + k dN_t
\]

(3)

\[
dX_t = [\Omega \Omega' + MX_t + X_t M'] dt + \sqrt{X_t} dB_t Q + Q' dB'_t \sqrt{X_t}
\]

(4)

- $X_t$ is a $(2 \times 2)$ symmetric, pos.def. matrix-valued process
- Interactions for $M, Q$ not diagonal
Properties (diffusive part)

stoch. volatility  \[ V_t := var\left(\frac{dS_t}{S_t}\right) = tr[X_t] = X_{11} + X_{22} \]

stoch. leverage effect  \[ cov\left(\frac{dS_t}{S_t}, dV_t\right) = 2 tr[ R'Q X_t ] \]

stoch. persistence  \[ \frac{1}{dt} E[dV_t] = tr[\Omega\Omega'] + 2 tr[MX_t] \]

Natural mapping to observable, economically important quantities (Karoui, Durrleman wp 2007)

level  \[ \sqrt{V_t} = \sqrt{tr[X_t]} \]

skew  \[ S_t = \frac{1}{2} \frac{tr[RQX_t]}{tr[X_t]^{3/2}} \]

term struct  \[ \mathcal{M}_t \approx \frac{1}{2} \frac{tr[MX_t]}{tr[X_t]^{1/2}} \]
Aim: **Separate** volatility effect from unspanned skewness/term structure

\[
\begin{pmatrix}
X_{11} & X_{12} \\
X_{12} & X_{22}
\end{pmatrix} = \begin{pmatrix}
\cos(\alpha) & \sin(\alpha) \\
-\sin(\alpha) & \cos(\alpha)
\end{pmatrix} \begin{pmatrix}
\nu_{1,t} & 0 \\
0 & \nu_{2,t}
\end{pmatrix} \begin{pmatrix}
\cos(\alpha) & -\sin(\alpha) \\
\sin(\alpha) & \cos(\alpha)
\end{pmatrix}
\]

\[
(X_{11}, X_{12}, X_{22}) \rightarrow (V_t, \xi_t, \alpha_t) \quad V_t = tr[X_t] = \nu_{1,t} + \nu_{2,t}; \quad \xi = \frac{\nu_{1,t}}{\nu_{1,t} + \nu_{2,t}}
\]

**Bounded:** $\xi[0,1]; \alpha[0,\pi]$

**Decompose** expressions of the type $tr[AX_t]$:

\[
Tr[AX_t] = \frac{V_t}{2} \left[ Tr(A) + (2\xi_t - 1) \left( \cos(2\alpha_t)(A_{11} - A_{22}) + \sin(2\alpha_t)(A_{12} + A_{21}) \right) \right]
\]

**Application:** Illustrate unspanned skewness/term structure components via an approximation of the short term volatility surface

\[
S_t \propto [RQX_t] \\
M_t \propto [MX_t]
\]
Illustration of state decomposition

Introduction
Empirical evidence
Model
Third factor
Properties
State decomposition
Illustration
Illustration (2)
Option pricing
Estimation
Performance
Stochastic
Coefficients
Conclusion
Illustration of state decomposition

Introduction
Empirical evidence
Model
Third factor
Properties
State decomposition
Illustration
Illustration (2)
Option pricing
Estimation
Performance
Stochastic
Coefficients
Conclusion
Illustration of state decomposition (2)
Option pricing with (affine) Laplace transform

\[ \Psi(\tau; \gamma) := E_t [\exp (\gamma Y_T)] = \exp \left( \gamma Y_t + tr [A(\tau)X_t] + B(\tau) \right) \] (5)

where \( A(\tau) = C_{22}(\tau)^{-1}C_{21}(\tau) \) with the \( 2 \times 2 \) matrices \( C_{ij}(\tau) \):

\[
\begin{pmatrix}
C_{11}(\tau) & C_{12}(\tau) \\
C_{21}(\tau) & C_{22}(\tau)
\end{pmatrix} = \exp \left[ \tau \begin{pmatrix}
M + \gamma Q'R & -2Q'Q \\
C_0(\gamma) & -(M' + \gamma R'Q)
\end{pmatrix} \right] \] (6)

\[ C_0(\gamma) = \frac{\gamma(\gamma - 1)}{2} I_2 + \Lambda \left[ (1 + \bar{k})^\gamma \exp \left( \gamma(\gamma - 1) \frac{\delta^2}{2} \right) - 1 - \gamma \bar{k} \right] \] (7)

\[ B(\tau) = \left\{ r - q + \lambda_0 \left[ (1 + \bar{k})^\gamma \exp \left( \gamma(\gamma - 1) \frac{\delta^2}{2} \right) - 1 - \gamma \bar{k} \right] \right\} \tau - \frac{\beta}{2} tr [\log C_{22}(\tau) - \tau(M' + R'Q)] \] (8)

See Leippold/Trojani wp 2008
Estimation strategy

  Challenging, avoid over-fitting
- Cross-section only; risk-neutral pricing
- Nested optimum
- Max. likelihood like Bates(2000), correct for heteroskedasticity

Parameter estimate

\[ \hat{\theta} = \arg \max_{\theta} - \frac{1}{2} \sum_{t} \left( \ln |\Omega_t| + e'_t \Omega_t^{-1} e_t \right) \] (9)

\( e_t(\theta, X_t^*(\theta)) \) = relative pricing error, \( \Omega_t \) = conditional cov. matrix of \( e_{i,t} \)

Implied state by NLS

\[ X_t^*(\theta) = \arg \min_{\{X_t\}} \left( \hat{C}_i(\theta, X_t) - C_i \right)^2 \] (10)
Performance

Pricing
Improvements 1
Improvements 2
Improvements 3
Dynamic properties
Eigenvectors/level
Eigenvectors/alpha
Eigenvectors/alpha
Stochastic
Coefficients

Conclusion
## Pricing

### In sample (2000-2004, monthly)

<table>
<thead>
<tr>
<th>State variables</th>
<th>$SV_{2,0}$</th>
<th>$SV_{3,0}$</th>
<th>$SV_{3,1}$</th>
<th>$SV_{J2,0}$</th>
<th>$SV_{J3,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$rmsE$</td>
<td>1.180</td>
<td>1.127</td>
<td>1.048</td>
<td>1.115</td>
<td>0.913</td>
</tr>
<tr>
<td>(stdv)</td>
<td>(0.370)</td>
<td>(0.348)</td>
<td>(0.285)</td>
<td>(0.446)</td>
<td>(0.324)</td>
</tr>
<tr>
<td>Within bid-ask</td>
<td>0.603</td>
<td>0.617</td>
<td>0.640</td>
<td>0.635</td>
<td>0.633</td>
</tr>
</tbody>
</table>

### Full sample (1996-09/2008)

<table>
<thead>
<tr>
<th>State variables</th>
<th>$SV_{2,0}$</th>
<th>$SV_{3,0}$</th>
<th>$SV_{3,1}$</th>
<th>$SV_{J2,0}$</th>
<th>$SV_{J3,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$rmsE$</td>
<td>1.937</td>
<td>2.057</td>
<td>1.570</td>
<td>1.862</td>
<td>1.457</td>
</tr>
<tr>
<td>(stdv)</td>
<td>(1.101)</td>
<td>(1.727)</td>
<td>(0.808)</td>
<td>(1.129)</td>
<td>(0.809)</td>
</tr>
<tr>
<td>$rmsIVE$</td>
<td>2.69</td>
<td>2.61</td>
<td>2.60</td>
<td>3.18</td>
<td>2.36</td>
</tr>
<tr>
<td>Within bid-ask</td>
<td>0.437</td>
<td>0.461</td>
<td>0.540</td>
<td>0.452</td>
<td>0.527</td>
</tr>
</tbody>
</table>
Improvements by pricing error of the 2-factor model

Introduction
Empirical evidence
Model
Performance
Pricing
- Improvements 1
- Improvements 2
- Improvements 3
Dynamic properties
Eigenvectors/level
Eigenvectors/alpha
Eigenvectors/alpha
Stochastic Coefficients
Conclusion

Improvement over $SV_{2,0}$

Improvement over $SV_{J2,0}$

Improvement $= \frac{\epsilon_{SV2,0} - \epsilon_{SV3,1}}{\epsilon_{SV2,0}}$
Improvements by volatility level

Introduction
Empirical evidence
Model
Performance
Pricing
Improvements 1
Improvements 2
Improvements 3
Dynamic properties
Eigenvectors/level
Eigenvectors/alpha

Stochastic
Coefficients
Conclusion

Improvement = \frac{\epsilon_{SV(J)2,0} - \epsilon_{SV(J)3,1}}{\epsilon_{SV(J)2,0}}

P. Gruber: Three make a dynamic smile
Improvements by model-implied $\alpha$

Introduction
Empirical evidence
Model
Performance
Pricing
Improvements 1
Improvements 2
Improvements 3
Dynamic properties
Eigenvectors/level
Eigenvectors/alpha
Eigenvectors/alpha
Stochastic Coefficients
Conclusion

Improvement over $SV_{2,0}$

$$\text{Improvement} = \frac{\epsilon_{SV(J)2,0} - \epsilon_{SV(J)3,1}}{\epsilon_{SV(J)2,0}}$$
### Dynamic properties

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Empirical evidence</th>
<th>Model</th>
<th>Performance</th>
<th>Pricing</th>
<th>Improvements 1</th>
<th>Improvements 2</th>
<th>Improvements 3</th>
<th>Dynamic properties</th>
<th>Eigenvectors/level</th>
<th>Eigenvectors/alpha</th>
<th>Eigenvectors/alpha</th>
<th>Stochastic Coefficients</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Eigenvectors

<table>
<thead>
<tr>
<th></th>
<th>$l$</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
<th>$T = 3206$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional</td>
<td>2</td>
<td>97.0</td>
<td>1.9</td>
<td>0.8</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>
Dynamic properties

<table>
<thead>
<tr>
<th>l</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional</td>
<td>2</td>
<td>97.0</td>
<td>1.9</td>
<td>0.8</td>
</tr>
<tr>
<td>$-0.44 &lt; \alpha/\pi \leq -0.15$</td>
<td>2</td>
<td>96.0</td>
<td>2.1</td>
<td>1.0</td>
</tr>
<tr>
<td>$T = 3206$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.15 &lt; \alpha/\pi \leq -0.09$</td>
<td>1</td>
<td>97.1</td>
<td>1.3</td>
<td>1.0</td>
</tr>
<tr>
<td>$T = 641$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.09 &lt; \alpha/\pi \leq -0.03$</td>
<td>1</td>
<td>97.1</td>
<td>1.7</td>
<td>0.7</td>
</tr>
<tr>
<td>$T = 641$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.03 &lt; \alpha/\pi \leq 0.02$</td>
<td>1</td>
<td>97.0</td>
<td>1.7</td>
<td>0.9</td>
</tr>
<tr>
<td>$T = 641$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.02 &lt; \alpha/\pi \leq 0.15$</td>
<td>2</td>
<td>96.2</td>
<td>1.9</td>
<td>1.3</td>
</tr>
<tr>
<td>$T = 641$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$l =$ significant components according to mean eigenvalue criterion.

($N = 56$, threshold=$\frac{1}{56}=1.79\%$)
Eigenvectors/level

Loading on PC 1

PC 2

PC 3

0.08 < level < 0.13

0.13 < level < 0.17

0.17 < level < 0.20

0.20 < level < 0.23

0.23 < level < 0.54

P. Gruber: Three make a dynamic smile
Eigenvectors/alpha ($SV_{3,1}$)
Eigenvectors/alpha ($SV_{J_{3,1}}$)
Stochastic Coefficient Interpretation
Interpret eigenvalues of $X_t$ as two volatility factors:

\begin{align*}
    d\nu_{1t} &= \left( \beta (\tilde{Q}'_t \tilde{Q}_t)^{11} + 2(\tilde{M}_t)^{11} \nu_{1t} + \frac{\nu_{1t}(\tilde{Q}'_t \tilde{Q}_t)^{22} + \nu_{2t}(\tilde{Q}'_t \tilde{Q}_t)^{11}}{\nu_{1t} - \nu_{2t}} \right) dt + 2\sqrt{\nu_{1t}(\tilde{Q}'_t \tilde{Q}_t)^{11}} d\nu_{1t} \\
    d\nu_{2t} &= \left( \beta (\tilde{Q}'_t \tilde{Q}_t)^{22} + 2(\tilde{M}_t)^{22} \nu_{2t} - \frac{\nu_{1t}(\tilde{Q}'_t \tilde{Q}_t)^{22} + \nu_{2t}(\tilde{Q}'_t \tilde{Q}_t)^{11}}{\nu_{1t} - \nu_{2t}} \right) dt + 2\sqrt{\nu_{2t}(\tilde{Q}'_t \tilde{Q}_t)^{22}} d\nu_{2t}
\end{align*}

$(\nu_1, \nu_2)'$ standard Brownian motion in $\mathbb{R}^2$

$\tilde{M}_t = O'_t M O_t$ and $\tilde{Q}_t = O'_t Q O_t$.

$O_t = \begin{pmatrix} \cos(\alpha_t) & -\sin(\alpha_t) \\ \sin(\alpha_t) & \cos(\alpha_t) \end{pmatrix}$
Factors $\mathcal{V}_{1t}$ and $\mathcal{V}_{2t}$ cannot cross (Wishart property),
**but** mean-reversion and vol-of vol *can*
Conclusion
### Conclusion

- Identified interacting + unspanned components in the volatility surface of S&P 500 index options.
- Matrix jump diffusion is a convenient framework for modeling interacting + unspanned factors.
- Estimated full matrix jump-diffusion model and nested models

**Find:**
- Three factors are indeed needed
- Better in and out-of sample fit
- Third factor should be interaction factor ($\alpha$)
- Largest improvements where 2 factor models are weak and $\alpha \neq 0$

- Appropriate conditioning provides evidence for a conditional two-factor structure $\rightarrow$ stochastic coefficient model.

### Future

- Use insights for more parsimonious models
- Apply to other fields of finance + economics
Spare slides
Short and long term expansion

Introduction
Empirical evidence
Model
Performance
Stochastic Coefficients
Conclusion
Short and long term expansion

Introduction
Empirical evidence
Model
Performance
Stochastic Coefficients
Conclusion

P. Gruber: Three make a dynamic smile
Short and long term expansion

Introduction
Empirical evidence
Model
Performance
Stochastic Coefficients
Conclusion
Numerical aspects

□ Nested optimization: very heavy computation

□ Major load: calculating the Laplace transform (not performing the Fourier inversion)

□ Matrix logarithm – use matrix rotation count algorithm

**Improve speed**

□ Optimized MATLAB code on a MATLAB cluster (32 cores)

□ Genetic optimization permits parallelizing parameter estimation

□ Cos-FFT (250 instead of 4096 evaluations of Laplace transform)

□ Separate evaluation of state-dependent and maturity-dependent parts of Laplace transform

□ Select a sample with few distinct maturities (monthly data, all Wednesdays)

□ Estimation still takes 1 week
Jumps

- Jump size like Bates: iid jumps

\[ \ln(1 + k) \sim N(\ln(1 + \bar{k}) - \frac{\delta^2}{2}, \delta^2) \]

- Jump intensity: extend Bates to matrix case

\[ \lambda_t = \lambda_0 + \Lambda_{11}X_{11} + \Lambda_{12}X_{12} + \Lambda_{22}X_{22} = \lambda_0 + tr[\Lambda X_t] \]

- Identification \( \rightarrow \Lambda \) upper triangular

- Ensure positive jump intensity:

\[
\begin{align*}
\Lambda_{11} &> 0 \\
\Lambda_{22} &> 0 \\
|\Lambda_{12}| &< 2\sqrt{\Lambda_{11}\Lambda_{22}}
\end{align*}
\]

- Unspanned jump intensity component