MULTIVARIATE EXTENSION OF PUT-CALL SYMMETRY

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Barrier-contingent claims

- $S_t = (S_{01}e^{t\lambda_1}e^{\xi_{t1}}, \ldots, S_{0n}e^{t\lambda_n}e^{\xi_{tn}}), \ t \in [0, T]$
  \((\lambda_1, \ldots, \lambda_n \text{ — deterministic carrying costs})\)

- $S_T = F\eta = (F_1\eta_1, \ldots, F_n\eta_n)$

**Barrier-contingent claim:**

$$X = f(S_T)\mathbb{I}\{\ldots\} = f(F\eta)\mathbb{I}\{\ldots\}$$

where $\mathbb{I}\{\ldots\}$ is the indicator of some barrier event and $f$ is some payoff function, e.g. $(k > 0)$

$$f(S_T) = (w_1F_1\eta_1 + \cdots + w_nF_n\eta_n - k)_+,$$

$$f(S_T) = (\max(w_1F_1\eta_1, \ldots, w_nF_n\eta_n) - k)_+,$$

$$f(S_T) = (w_1F_1\eta_1 + \cdots + w_nF_n\eta_n)_+.$$
Well-known classic European put-call symmetry (holding for certain models)

\[ E(F\eta - k)_{+} = E(F - k\eta)_{+} \quad \text{for every } k \geq 0. \]

In view of that, consider

- \( \eta = (\eta_1, \ldots, \eta_n), (1, \eta_1, \ldots, \eta_n) \) — random price changes
- \( f(\eta) \) — payoff function (forward prices are included in the payoff functions)
- Discussion: In which case is \( E_Q f(\eta) \) invariant with respect to swaps of its arguments (expectation w.r.t. martingale measure)?

**Main application:** Semi-static hedging of certain barrier-contingent claims, i.e. the replication of these contracts by trading European-style claims at no more than two times after inception.
Some historic remarks

- *Bates’ rule:*

- *Semi-static hedge* of barrier options (based on J. Bowie and P. Carr):
  (a call option at the barrier can be converted in certain put options)

- *Lévy markets:*

- *Multiasset case:*
**Duality principle alone does not suffice**

For the duality principle, see Eberlein, Papapantoleon & Shiryaev 2008, 2009 and the literature cited therein.

Since $\mathbb{E}_Q \eta = 1$, define

$$\frac{d\tilde{Q}}{dQ} = \eta.$$ 

With $\tilde{\eta} = \eta^{-1}$

$$\mathbb{E}_Q (H\eta - k)_+ = \mathbb{E}_Q \eta^{-1} (H\eta - k)_+ = \mathbb{E}_{\tilde{Q}} (H - k\tilde{\eta})_+$$

$$= kH^{-1} \mathbb{E}_{\tilde{Q}} (H^2 k^{-1} - H\tilde{\eta})_+.$$ 

Need

$$\mathbb{E}_Q (H\eta - k)_+ = kH^{-1} \mathbb{E}_Q (H^2 k^{-1} - H\eta)_+$$

(resp. equivalent properties) for symmetry based semi-static hedges.
Most important multivariate functions

- Basket option $\mathbb{E}_Q \left( u_0 + u_1 \eta_1 + \cdots + u_n \eta_n \right)_+$
  function of $(\eta_0 = 1, \eta_1, \ldots, \eta_n)$

- Calls (puts) on maximum/minimum, e.g.
  $$\mathbb{E}_Q \left( \max(u_1 \eta_1, \ldots, u_n \eta_n) - u_0 \right)_+$$

  for our symmetry analysis can be replaced by
  $$\mathbb{E}_Q \max(u_0, u_1 \eta_1, \ldots, u_n \eta_n)$$

- Exchange option $\mathbb{E}_Q \left( u_1 \eta_1 + \cdots + u_n \eta_n \right)_+$
Characterisation of distributions

- Breeden & Litzenberger (1978): the prices of all call (resp. put) options determine the distribution of the single underlying.

- The prices of all basket options determine the multiasset distribution
  Carr & Laurence — absolutely continuous case;
  the general case is implicit in Henkin & Shananin, Koshevoy & Mosler.
• The same holds for all options on the maximum (weighted)
  \[ \max(u_0, u_1 \eta_1, \ldots, u_n \eta_n) \] or minimum \[ \min(u_0, u_1 \eta_1, \ldots, u_n \eta_n) \].

• The same holds for calls (puts) on maximum/minimum, e.g.
  \[ (\min(u_1 \eta_1, \ldots, u_n \eta_n) - u_0)^+ . \]

*Does not hold for exchange options* \( (u_1 \eta_1 + \cdots + u_n \eta_n)_+ \).
Information in exchange options

Let $\eta = e^\xi$ and $\eta^* = e^{\xi^*}$ be integrable random vectors. Then

$$E(\langle u, \eta \rangle)_+ = E(\langle u, \eta^* \rangle)_+ \quad \text{for all } u \in \mathbb{R}^n$$

if and only if

$$\varphi_\xi(u - \iota w) = \varphi_{\xi^*}(u - \iota w) \quad (1)$$

for all $u \in \mathbb{H}$, where

$$\mathbb{H} = \{ u \in \mathbb{R}^n : \sum_{k=1}^{n} u_k = 0 \},$$

and for at least one (and then necessarily for all) $w$, such that $\sum w_i = 1$ and both sides in (1) are finite.

Infinitely divisible case: (1) can be expressed via the Lévy triplet.
Consequences

• Prices of all basket options determine the prices of all European options (depending on the same assets, with the same maturity).

• Prices of all exchange options determine them for a certain class of payoff functions.
Symmetries of multivariate option prices functions

• Basket option \( E_Q \left( u_0 + u_1 \eta_1 + \cdots + u_n \eta_n \right)_+ \)
  (swap \( u_0 \) and \( u_i \)) — \( \eta \) is \( i \)-self-dual (for all \( (u_0, u) \in \mathbb{R}^{n+1} \))

• Option on the maximum \( E_Q \max(u_0, u_1 \eta_1, \ldots, u_n \eta_n) \)
  (swap \( u_0 \) and \( u_i \)) — \( \eta \) is \( i \)-self-dual (for all \( (u_0, u) \in \mathbb{R}^{n+1} \))

• Exchange option \( E_Q \left( u_1 \eta_1 + \cdots + u_n \eta_n \right)_+ \)
  (swap \( u_i \) and \( u_j \) with \( u_0 = 0 \)) — \( \eta \) is \( i,j \)-swap-invariant
  (for all \( u \in \mathbb{R}^n \))
Characterisation of self-dual distributions

Integrable \( \eta \) is \( i \)-self-dual if and only if e.g.

- \( \mathbb{E} f(\eta) = \mathbb{E}[f(\kappa_i(\eta))\eta_i] \) for all integrable payoffs \( f \), where

\[
\kappa_i(x) = \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{1}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right).
\]

- The distribution of \( \eta \) under \( Q \) coincides with the distribution of \( \kappa_i(\eta) \) under \( Q^i \), where

\[
\frac{dQ^i}{dQ} = \eta_i.
\]

- If \( \eta \) is absolutely continuous,

\[
p_\eta(x) = x_i^{-n-2}p_\eta(\kappa_i(x)) \text{ a.e.}
\]
• Characterisation in terms of the distribution of \( \xi = \log \eta \)

\[
\varphi_\xi\left(u - \frac{1}{2} e_i\right) = \varphi_\xi\left(K_i^\top u - \frac{1}{2} e_i\right), \quad u \in \mathbb{R}^n,
\]

where

\( K_i x = (x_1 - x_i, \ldots, x_{i-1} - x_i, -x_i, x_{i+1} - x_i, \ldots, x_n - x_i) \),

(some other equivalent complex shifts are also possible).

Infinitely divisible case: This characterisation can be expressed via the Lévy triplet.
PCS in the one asset case

• Classic European put-call symmetry is equivalent to many other definitions.

• Almost any tail behaviour is possible.

• $\eta$ has a non-negative skewness and for infinitely divisible $\xi = \log \eta$, $\xi$ has non-positive skewness.

• For much more, see Carr and Lee 2009 and the literature cited therein.
Swap-invariance and PCS

Integrable $\eta$ is called $ij$-swap-invariant if

$$E_Q(u_1\eta_1 + \cdots + u_n\eta_n)^+, \ u \in \mathbb{R}^n,$$

is $\pi_{ij}$-invariant (swap $u_i$ and $u_j$).

Integrable $\eta$ is $ij$-swap-invariant if and only if the $(n - 1)$-dimensional random vector

$$\tilde{\kappa}_j(\eta) = \left( \frac{\eta_1}{\eta_j}, \ldots, \frac{\eta_{j-1}}{\eta_j}, \frac{\eta_{j+1}}{\eta_j}, \ldots, \frac{\eta_n}{\eta_j} \right)$$

is self-dual with respect to the $i$th component under $Q^j$. 

Characterisation

An integrable random vector $\eta = e^\xi$ is $ij$-swap-invariant if and only if the characteristic function of $\xi$ satisfies

$$\varphi_\xi(u - \frac{1}{2} e_{ij}) = \varphi_\xi(\pi_{ij} u - \frac{1}{2} e_{ij})$$

for all

$$u \in H = \{ u \in \mathbb{R}^n : \sum_{k=1}^{n} u_k = 0 \},$$

where $e_{ij} = e_i + e_j$ (many equivalent complex shifts).

Infinitely divisible case: This characterisation can be expressed via the Lévy triplet.
Examples

- **Black-Scholes** case: *Each bivariate* risk-neutral log-normal distribution is swap-invariant, no matter what volatilities of the assets and correlation are.
- The considerable effective degrees of freedom for modelling two assets based on dependent **generalised hyperbolic Lévy processes** only slightly decrease if we ensure that the bivariate swap-invariance property holds.
- Etc.
Example: Certain knock-out Margrabe \( (n = 2) \)

- **Payoff**

\[
X_{sw} = (S_{T1} - S_{T2}) + \mathbb{I}_{c > \frac{S_{t2}}{S_{t1}}} \forall t \in [0, T]
\]

with \( c \geq 1, 0 < \frac{S_{02}}{S_{01}} < c \), and (for simplicity) assume

\[
(S_{t1}, S_{t2}) = (S_{01} e^{\lambda t} e^{\xi_{t1}}, S_{02} e^{\lambda t} e^{\xi_{t2}}), (\xi_{t1}, \xi_{t2}), t \in [0, T],
\]

is a Brownian motion with drift and non singular covariance matrix

\[
\mu = -\left( \frac{\sigma_1^2}{2}, \frac{\sigma_2^2}{2} \right) \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}.
\]

- **Hedge portfolio:**
  - long position in the Margrabe option with payoff function
    \[
    (S_{T1} - S_{T2})_+,
    \]
  - short position in the weighted Margrabe option with payoff function
    \[
    (c^{-1}S_{T2} - cS_{T1})_+.
    \]
Verification of the hedge

• If the barrier is not hit, then \( cS_{t_1} > S_{t_2} \) for all \( t \); the short position
  \( (c^{-1}S_{T_2} - cS_{T_1})_+ \) expires worthless and the long position
  \( (S_{T_1} - S_{T_2})_+ \) replicates the option.

• If \( cS_{\tau_1} = S_{\tau_2} \), then the values of these two options at time \( \tau \) are
  identical.
Problems with carrying costs

Write
\[ e^\lambda \eta = (e^{\lambda_1 + \xi_1}, \ldots, e^{\lambda_n + \xi_n}), \]
where \( \lambda_i = r - q_i \) (\( q_i \)-dividend yield), \( i = 1, \ldots, n \) (and for simplicity of notation \( T = 1 \)).

The problem in self-dual cases

- For applications usually \( \mathbb{E} e^{\xi_j} = 1, j = 1, \ldots, n \).
- Multiplication by \( e^{\lambda_i}, \lambda_i \neq 0 \), moves the expectation away from one.
- \( e^{\lambda + \xi} \) self-dual with respect to the \( i \)th coordinate \( \Rightarrow \mathbb{E} e^{\lambda_i + \xi_i} = 1 \).
- For semi-static hedging, symmetry is rather needed in \( e^{\lambda + \xi} \) than in \( e^\xi \).
Quasi-self-duality

$\eta = e^\xi$ is quasi-self-dual (with respect to the $i$th coordinate) if there exist $\lambda \in \mathbb{R}^n$ and $\alpha \neq 0$ such that $(e^{\lambda+i}\xi)^\alpha$ is integrable and self-dual with respect the $i$th coordinate.

Univariate power-transform: Carr and Lee (2009), based on earlier work of Carr and Chou.

For the multivariate case

$$E[f(S_T)]$$

$$= E\left[f\left(\frac{S_{0i}}{S_{Ti}}(S_{T1}, \ldots, S_{T(i-1)}, S_{0i}, S_{T(i+1)}, \ldots, S_{Tn})\right)\left(\frac{S_{Ti}}{S_{0i}}\right)^\alpha\right],$$

eqc

A similar extension to quasi-swap-invariance is known (useful for non-equal carrying costs).
Finding $\alpha$ in infinitely divisible cases

To ensure that $\mathbb{E} \eta_i = 1$ the value $\alpha$ must satisfy

$$a_{ii} \alpha = a_{ii} - 2\lambda_i + 2 \int_{\mathbb{R}^n} (e^{x_i} - 1 - x_i e^{\frac{\alpha}{2} x_i} 1_{\|x\| \leq 1}) d\nu(x),$$

where $\|x\|^2 = \frac{1}{2} (\|x\|^2 + \|K_i x\|^2)$.

Usually not easy to solve (even for $n = 1$) and solution(s) may not exist.

There are some friendly special cases.
References


See also

  Math. Finance

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