Pricing algorithm for swing options based on Fourier Cosine Expansions

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Details of the swing option
  ▶ Contract details
  ▶ Pricing details

Fourier Cosine algorithm for swing options
  ▶ Recovery time—the penalty time between two consecutive exercises—plays an important role.

Numerical results of option contracts varying in recovery time and upper bound of exercise.
Swing options give contract holders the right to modify amounts of future delivery of certain commodities, such as electricity or gas.

We deal with an American style swing options which can be exercised at any time before expiry and more than once, with the following restrictions:

- **Recovery time** between two consecutive exercises ($\tau$).
  With exercise amount $D$ we have $\tau_D = C$ or $\tau_D = f(D)$
- **Upper bound** of exercise amount $|D|$: $|D| < L$
Payoff of swing option

Payoff of a swing option with varying $S$ and $D$ reads

$$g(S, T, D) = D \cdot (\max(S - K_a, 0) - \max(S - S_{max}, 0) + \max(K_d - S, 0) - \max(S_{min} - S, 0)),$$

**Figure:** Example of a payoff of a swing option with $S_{min} = 20$, $K_d = 35$, $K_a = 45$, and $S_{max} = 80$, and $S$ and $D$ varying.
Recovery time $\tau_R(D)$ is assumed to be an increasing function of exercise amount $D$. The shortest recovery time is when we only exercise one amount of the swing option.

- If $T - t < \tau_R(1)$, it is impossible to exercise more than once before expiry. If profitable to exercise, then exercise at $D_{\text{max}} = L$ amount. Therefore we are dealing with an American type option which reads at each step

$$v(s, t) = \max(g(s, t, L), c(s, t))$$
If \( T - t \geq \tau_R(1) \), there exists multiple exercise opportunities before expiry. Apart from the optimal exercise time we also need to find the optimal exercise amount at each time step:

\[
\nu(s, t) = \max_D \left( \max(g(s, t, D) + \phi'_D(s, t), c(s, t)) \right)
\]

Here \( g(s, t, D) \) is the instantaneous profit obtained from the exercise of a swing option and \( \phi'_D \) is the continuation value from \( t + \tau_R(D) \).
Option pricing based on Fourier Cosine expansions

Truncating the infinite integration range of the Risk-Neutral formula

\[ v(x, t_0) = e^{-r \Delta t} \int_a^b v(y, T)f(y|x)dy \]

The conditional density function of the underlying is approximated as follows:

\[ f(y|x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1} \text{Re}(\phi(\frac{k\pi}{b-a}; x) \exp(-i\frac{ak\pi}{b-a})) \cos(k\pi \frac{y-a}{b-a}), \]

Replacing \( f(y|x) \) by its approximation and interchanging integration and summation, we obtain the **COS algorithm** for option pricing

\[ v(x, t_0) = e^{-r \Delta t} \sum_{k=0}^{N-1} \text{Re}(\phi(\frac{k\pi}{b-a}; x)e^{-ik\pi \frac{a}{b-a}})V_k \]

where \( V_k \) is the Fourier Cosine coefficient of option value \( v(y, T) \).
The swing option is equivalent to an American option, which is can be obtained from **Bermudan option values** with different numbers of exercise dates, i.e. a 4–point repeated Richardson extrapolation.

**Pricing algorithm**

- **Initialization:** Compute $V_k(t_M)$ at $t_M = T$.
- **Backward recursion:** For $m = M - 1, \cdots, 1$, recover
  \[
  V_k(t_m) = \frac{2}{b-a} \int_a^b v(x, t_m) \cos(k \pi \frac{x-a}{b-a}) \, dx
  \]
  from $V_k(t_{m+1})$, where
  \[
  v(x, t_m) = \max(g(x, t_m), c(x, t_m)).
  \]
- **Last step:**
  \[
  v(x, t_0) = e^{-r(t_1-t_0)} \sum_{k=0}^{N-1} \text{Re} \left( \phi \left( \frac{k \pi}{b-a}; x \right) e^{-ik \pi \frac{a}{b-a}} \right) V_k(t_1)
  \]

In our implementation we set $x = \log(s)$.
At expiry

At $t_M = T$ option value $v$ equals the payoff $g$ and we have for the Fourier cosine coefficients of the swing option value:

$$V_k(t_M) = G_k(a, \ln(K_d), L) + G_k(\ln(K_a), b, L),$$

where

$$G_k(x_1, x_2, L) = \frac{2}{b - a} \int_{x_1}^{x_2} g(x, t_M, L) \cos(k\pi \frac{x - a}{b - a}) dx$$

is the Fourier cosine coefficient of the swing option payoff which has analytic solution.
Backward Recursion

At each time step $t_m, m = M - 1, \ldots, 1$

$$V_k(t_m) = \frac{2}{b - a} \int_a^b v(x, t_m) \cos(k \pi \frac{x - a}{b - a}) dx$$

where $v(x, t_m) = \max(g(x, t_m), c(x, t_m))$. We identify the regions where $v = c$ and those where $v = g$ and split $V_k$ accordingly.

By Newton’s method we find the early exercise points where $c = g$. For swing options there are two early exercise points, $x_d^m$ and $x_a^m$.

$V_k(t_m)$ can be split as

$$V_k(t_m) = G_k(a, x_d^m, D) + C_k(x_d^m, x_a^m, t_m) + G_k(x_a^m, b, D),$$

where $C_k$ and $G_k$ are the Fourier cosine coefficients of the continuation value and swing option payoff.
Calculation of $G_k$ and $C_k$

At each time step $t_m, m = M - 1, \cdots, 1$, we have

- $G_k$ has analytic solution with computation complexity $O(N)$.
- $C_k$ can be rewritten as a matrix-vector product representation:

$$C(x_1, x_2, t_m) = \frac{e^{-r\Delta t}}{\pi} \text{Im}\{(M_c + M_s)u\},$$

For Lévy processes the matrices $M_s$ and $M_c$ have a Toeplitz and Hankel structure, respectively and $C_k$ can be calculated with the help of the Fast Fourier Transform, with computation complexity $O(N\log_2 N)$. For other processes, $C_k$ is calculated with computation complexity of $O(N^2)$. 

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Pricing algorithm for $t : T - t \geq \tau_R(1)$

In the interval $\{ t : T - t > \tau_R(1) \}$ the swing option can be exercised more than once before expiry and recovery time plays an important role. In this case we have

$$ v(x, t) = \max_D \left( \max g(x, t, D) + \phi^t_D(x, t), c(x, t) \right) $$

It is an American-style option with recovery time and multiple exercise opportunities.

- Due to recovery time, the payoff also includes $\phi^t_D(x, t)$, the continuation value from $t + \tau_R(D)$.
- Due to multiple exercise opportunities, we take the maximum over the resulting payoff for all possible values of $D$, and the continuation value from the previous time step.
Backward Recursion

With

- \( A_D, D = 1, \cdots, L \) is the regions in which exercising the swing option with \( D \) commodity units results in the highest profit
  \[ g(x, t_m, D) + \phi^t_m(x, t_m). \]
- \( A_c \) is the region in which \( c(x, t) \) is the maximum. In other words, with the commodity price in \( A_c \), it is profitable not to exercise the swing option.

Then for \( m = M - 1, \cdots, 1 \),

\[
V_k(t_m) = \frac{2}{b-a} \int_{A_c} c(x, t_{m+1}) \cos\left(\frac{k\pi(x-a)}{b-a}\right) dx \\
+ \sum_{D=1}^{L} \int_{A_D} g(x, t_m, D) + \phi^t_m(x, t_m) \cos\left(\frac{k\pi(x-a)}{b-a}\right) dx
\]

And \( v(x, t_0) = e^{-r(t_1-t_0)} \sum_{k=0}^{N-1} \text{Re} \left( \phi\left(\frac{k\pi}{b-a}; x\right) e^{-i k \pi \frac{a}{b-a}} \right) V_k(t_1) \)
Calculation of $V_k$

At each time step $t_m$, $m = M - 1, \cdots, 1$, $V_k(t_m)$ can be rewritten as:

$$V_k(t_m) = \frac{2}{b-a} \left( \int_{A_c} c(x, t_{m+1}) \cos \left( \frac{k\pi(x-a)}{b-a} \right) dx \right)$$

$$+ \sum_{D=1}^{L} \int_{A_D} g(x, t_m, D) \cos \left( \frac{k\pi(x-a)}{b-a} \right) dx$$

$$+ \sum_{D=1}^{L} \int_{A_D} \phi_D^{t_m}(x, t_m) \cos \left( \frac{k\pi(x-a)}{b-a} \right) dx$$

$$\triangleq V_c + V_g + V_\phi$$

- $A_D$, $D = 1, \cdots, L$, and $A_c$ are determined by Newton’s method.
- $V_c$ and $V_g$ are calculated the same way as $G_k$ and $C_k$.
- $V_\phi$ is calculated similarly as $G_c$, but from $V_k(t_m + \tau_R(D))$ instead of $V_k(t_{m+1})$. This implies we need to store intermediate values of $V_k$.

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Constant recovery time

In this case additional profit is not connected to an extra penalty. We have either $D = L$ or $D = 0$. Two early-exercise points $x_m^d$ and $x_m^a$ are to be determined, so that

$$c(x_m^d, t_m) = g(x_m^d, t_m, L) + \phi_L^{t_m}(x_m^d, t_m),$$

and

$$c(x_m^a, t_m) = g(x_m^a, t_m, L) + \phi_L^{t_m}(x_m^a, t_m),$$

And $V_k(t_m)$ is split into three parts,

$$V_k(t_m) = G_k(a, x_m^d, L) + C_k(x_m^d, x_m^a, t_m) + G_k(x_m^a, b, L).$$
Numerical results

We discuss two types of recovery time functions:

- **Constant recovery time**: If $D \neq 0$, we set $\tau_R(D, t) = \frac{1}{4}$. In other words, the option holder needs to wait three months between two consecutive swing actions, independent of the time point of exercise or the size $D$.

- **State-dependent recovery time**: We assume $\tau_R(D, t) = \frac{D}{12}$ which implies that if the option holder exercises the swing option with $D$ units, he/she has to wait $D$ months before the option can be exercised again.

In our numerical examples presented here, the underlying follows the CGMY model (exponential Lévy process) with $Y = 1.5$. 
Convergence over $M$ and estimation of American option

Two approximation methods are compared:

- Direct approximation: Bermudan-style options with $M = N/2$.
- Richardson 4-point extrapolation technique.

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<th>$n = \log_2 N$</th>
<th>$P(N/2)$ option value</th>
<th>$P(N/2)$ CPU time</th>
<th>Richardson option value</th>
<th>Richardson CPU time</th>
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<td>203.27</td>
<td>137.390</td>
<td>13.21</td>
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Table: Convergence over $M$ and comparison between two approximation methods for American-style swing option, with $t = T - 0.5$, $\tau_R(D) = 0.25$, $C = 1$, $G = 5$, $M = 5$, $Y = 1.5$
American style swing option value

**Figure:** American-style swing option values under the CGMY processes with constant recovery time, $\tau_R(D) = 0.25$.

Jumps are observed at $T - t = 0.25$, $T - t = 0.5$ and $T - t = 0.75$, where the maximum number of remaining exercise possibilities is reduced by 1.
Swing contracts with varying flexibility

(a) Varying upper bound $L$  
(b) Varying recovery time $\tau_R(D)$

Figure: CGMY process, $T - t = 1$; Left: Different values for $L$, and fixed $\tau_R(D, t) = \frac{1}{12} D$; Right: Different Recovery time, and fixed $L = 5$.

- Higher values of $L$ give rise to higher option values.
- Longer recovery time gives lower option prices
The optimal exercise amount $D_{opt}$

Below is a figure of $D_{opt}$ over different underlying prices, with $\tau_R(D) = \frac{1}{12} D$.

- As $S$ goes beyond $K_d$ and $K_a$, $D_{opt}$ tends to increase, because in this region instantaneous profit $g(x, t, D)$ tends to dominate in the payoff $g(x, t, D) + \phi^+_D(x, t)$.

- Between $S = 20$ and $S = 25$, $D_{opt} = 0$, since $g(x, t, D) = 0$ for all $D > 0$ in this interval.
Convergence of the algorithm

With $N$ the number of Fourier Cosine expansion terms, and $L$ the upper bound of exercise amount,

<table>
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<th>N</th>
<th>256</th>
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- With $N = 256$ the swing option algorithm reaches basis point accuracy.
- The algorithm is flexible regarding the variation in parameter $L$. 
Conclusions

- We presented an efficient pricing algorithm for swing options with early-exercise features, based on Fourier Cosine Expansions.
- It performs well for different swing contracts with varying flexibility in upper bounds of exercise amount and different recovery times.
- For Lévy processes the Fast Fourier Transform can be applied in the backward recursion procedure, which gives us Bermudan-style swing option prices accurate to one basis point in milli-seconds for constant recovery time, and in less than one, to three seconds for dynamic recovery time with different values of $L$.
- The Richardson 4-point extrapolation technique is efficient in pricing American-style swing options.